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Abstract

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THE DIRICHLET PROBLEM FOR AN EQUATION WITH THE MONGE-AMPÈRE OPERATOR

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In the paper (1), one of the authors of the present article established that, for an equation of the form

$$rt - s^2 = \varphi(x, y, z, p, q), \qquad \varphi \geqslant 0, \tag{1}$$

when the function φ grows sufficiently rapidly as $p^2 + q^2 \to +\infty$, the Dirichlet problem, generally speaking, has no solutions if the satisfaction of the boundary condition is understood in the classical sense. The simplest example of this is the Dirichlet problem in the disk $K : x^2 + y^2 \leqslant a^2 < 1$:

$$rt - s^2 = (1 + p^2 + q^2)^2; \qquad z\big|_{\mathrm{fr}\,K} = bx.$$

For sufficiently large values, in absolute value, of the number b, this problem has no solutions.

In the paper (2) it was proved that if the domain Ω is bounded by a closed convex curve Γ with substantially positive specific curvature, and the function $\varphi(x, y, z, p, q)$ is continuous and satisfies the inequalities

$$0 \leqslant \varphi(x, y, z, p, q) \leqslant c_0(1 + p^2 + q^2), \qquad c_0 = \mathrm{const}, \tag{2}$$

then the Dirichlet problem for equation (1) has a generalized solution (see (1,2)), which assumes on Γ a prescribed continuous function. Using other considerations, A. V. Pogorelov in (3) established that the indicated problem is solvable also in the case when the function φ does not increase with respect to z and satisfies the condition, less restrictive than (2),

$$0 \leqslant \varphi(x, y, z, p, q) \leqslant c_0(1 + p^2 + q^2)^{3/2}.$$

In the present article conditions are established under which the Dirichlet problem for the equation $rt - s^2 = \varphi(x, y, z, p, q)$ is solvable if the function

$\varphi(x, y, z, p, q)$ has an arbitrary power order of growth in p, q, or $p^2 + q^2 \to +\infty$, and the specific curvature of the curve Γ may tend to zero. We restrict ourselves to the case of power growth of the function φ in p, q only for the sake of simplicity of exposition. The methods which we use below, as is easy to see, apply to functions $\varphi(x, y, z, p, q)$ having arbitrary growth in p, q as $p^2 + q^2 \to +\infty$.

Let $R(p, q)$ be a positive function continuous on the entire p, q-plane; we shall assume throughout that there exist positive constants c_0 and k such that, for all p, q, the inequality

$$R(p, q) \leqslant c_0 (1 + p^2 + q^2)^k \tag{3}$$

holds.

Let P be a convex surface that projects one-to-one onto the bounded open convex domain Ω. By $\omega(1/R, P, H)$ we shall denote its conditional curvature generated by the function $1/R(p, q)$ (^{1}).

We shall say that the specific curvature of the convex curve Γ (Γ is the boundary of the domain Ω) at a point Q_0 has order of vanishing not greater than ν ($\nu \geqslant 0$) if there exists such a positive constant h that

$$\theta / l \geqslant h l^{\nu}, \tag{4}$$

where l is the length of an arbitrary sufficiently small arc of the curve Γ containing the point Q_0, and θ is the angle between the supporting lines to Γ drawn at the end-

ends of this arc. Below we shall constantly assume that the curve Γ has, at every point, order of extension not greater than ν, where $\nu \geqslant 0$ is a constant number.

Theorem 1. Let surfaces $P_1, P_2, \ldots, P_n, \ldots$, convex in the direction $z > 0$ ($z < 0$), be projected one-to-one onto convex domains $\Omega_n \subseteq \Omega$, and, as $n \to \infty$, converge to a surface P which is projected one-to-one inside Ω. Let the boundary curves $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ of these surfaces converge to a curve γ, projected one-to-one onto Γ. Suppose, furthermore, that there exist numbers $\lambda \geqslant 0$, $a > 0$ such that for every point $Q_0 \in \Gamma$ there is a neighborhood S, for all Borel sets of which lying entirely in Ω, the inequality holds

$$\varlimsup_{n \to \infty} \omega\left(\frac{1}{R}, P_n, H \cap \Omega_n\right) \leqslant a \left[\sup_{(x,y) \in H} \rho(x, y)\right]^{\lambda} \operatorname{mes} H * . \tag{5}$$

Then, if the numbers λ, k, ν are related by the inequality

$$k \leqslant 1 + \frac{1}{\nu + 2} + \frac{\lambda}{2}, \tag{6}$$

the curve γ is the boundary of the surface P.

From this theorem there follow the following two theorems, important for further applications.

Theorem 2. Let the conditions of Theorem 1 be fulfilled and let $\Omega_n = \Omega$ for all n. Then the sequence of convex functions $z_n(x, y)$, for which the surfaces P_n are graphs, converges uniformly in $\Omega + \Gamma$ to the function $z(x, y)$ defining the surface P in Ω.

Theorem 3. Let $\mu(H)$ be a completely additive nonnegative set function on the domain Ω, satisfying the following conditions: there exist constants $\lambda \geqslant 0$, $a > 0$ such that for every point $Q_0 \in \Gamma$ there is a neighborhood S, for all Borel sets H of which lying entirely in Ω, the inequality

$$\mu(H) \leqslant a \left[\sup_{(x,y)\in H} \rho(x, y) \right]^{\lambda} \operatorname{mes} H, \qquad \mu(\Omega) > \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp\, dq}{R(p, q)} \tag{7}$$

is valid ($\rho(x, y)$ is the distance from the point (x, y) to Γ). Then, whatever closed curve γ, having a one-to-one projection onto Γ, may be, there exists a unique surface P, convex in the direction $z > 0$ ($z < 0$), with boundary γ, which is projected one-to-one onto $\Omega + \Gamma$ and satisfies the equation

$$\omega(1/R, P, H) = \mu(H).$$

Let us consider in particular the case when

$$\mu(H) = \iint_H \varphi(x, y)\, dx\, dy;$$

the function $\varphi \geqslant 0$ is summable. If there exist constants $\lambda \geqslant 0$ and $a > 0$ such that for all points (x, y) sufficiently close to Γ the inequality

$$\varphi(x, y) \leqslant a[\rho(x, y)]^{\lambda} \tag{8}$$

is valid, then the set function $\mu(H)$ satisfies conditions (7). Hence, from Theorem 3 it follows that the equation $rt - s^2 = \varphi(x, y)R(p, q)$ has a unique generalized solution (see (1)), taking on Γ any prescribed continuous function of a point of Γ and having convexity directed toward $z > 0$ ($z < 0$).

Denote by W^+ (W^-) the set of all functions convex in the direction $z > 0$ ($z < 0$), defined in Ω. Let $\psi(X)$ be a continuous function of a point X of the curve Γ. A solution of the boundary-value problem

$$rt - s^2 = \varphi(x, y)R(p, q), \qquad Z|_{\Gamma} = \psi(X)$$

* $\rho(x, y)$ is the distance from the point (x, y) to Γ.

in the class W^+, if it exists, by $A_R\varphi$. Put, for $u \in [0, +\infty)$,

$$M(u) = \iint_{p^2+q^2 \le (u/d)^2} \frac{dp\,dq}{R(p,q)}$$

(d is the diameter of Ω). In (^{1}) it is proved that, if $A_R\varphi$ exists, then

$$M\left(\max_{\Omega+\Gamma} A_R\varphi - \max_{\Gamma} \psi(x)\right) \le \iint_{\Omega} \varphi(x,y)\,dx\,dy \le M(+\infty).$$

Theorem 4. Let $\Phi = \{\varphi(x,y)\}$ be some family of nonnegative functions, summable in Ω, satisfying condition (8) with the same constants a, λ for all $\varphi \in \Phi$. Suppose further that

$$I_\Phi = \sup_{\varphi\in\Phi} \iint_{\Omega} \varphi(x,y)\,dx\,dy < M(+\infty).$$

Then the operator A_R maps the family Φ into a set of class W^+ compact in the sense of uniform convergence.

Theorem 5. If Φ is a family of functions satisfying the condition of Theorem 4, and the sequence $\varphi_n \in \Phi$ is such that the set functions

$$\mu_n(H) = \iint_H \varphi_n(x,y)\,dx\,dy$$

converge weakly to the set function

$$\mu_0(H) = \iint_H \varphi_0(x,y)\,dx\,dy$$

inside Ω, then the sequence of functions $A_R\varphi_n$ converges uniformly to the function $A_R\varphi_0$.

We now pass to the consideration of the Dirichlet problem for the equation

$$rt - s^2 = \varphi(x,y,z,p,q).$$

Without loss of generality, it will be more convenient for us to write this equation in the form

$$rt - s^2 = R(p,q)f(x,y,z,p,q).$$

The factor $R(p,q)$, roughly speaking, carries the character of growth of the function $\varphi(x,y,z,p,q)$ as $p^2+q^2 \to +\infty$. With respect to the domain Ω and the function $R(p,q)$ we retain the assumptions made at the beginning of the paper.

We shall assume the function $f(x, y, z, p, q)$ to satisfy the following conditions: a) it is defined, continuous, and nonnegative for $0 \le z < R_0$, $(x, y) \in \Omega$, $-\infty < p, q < +\infty$ (in the particular case R_0 may also be $+\infty$); b) for every $R \in [0, R_0)$ there exist constants $a > 0$, $\lambda \ge 0$, $\varepsilon > 0$ such that, for all $(x, y) \in \Omega$ satisfying the condition $\rho(x, y) < \varepsilon$, the inequality

$$f(x, y, z, p, q) \le a[\rho(x, y)]^{\lambda}$$

holds for arbitrary p, q and $z \in [0, R]$; here it is assumed that the condition

$$k \le 1 + \frac{1}{\nu + 2} + \frac{\lambda}{2};$$

is fulfilled; c) for all $R \in [0, R_0)$ the inequality

$$F(R) = \iint_{\Omega} f_R(x, y)\,dx\,dy < M(+\infty), \qquad \text{where} \qquad f_R(x, y) = \sup_{\substack{0 \le z \le R \\ -\infty < p, q < +\infty}} f(x, y, z, p, q).$$

We shall assume that the function $\psi(x)$ defining the boundary condition in the Dirichlet problem satisfies the inequalities $0 \le \psi(x) < R_0$.

Denote by $\overline{W}^{+}_{\psi,R}$ the set of all functions in $\overline{W}^{+}$ whose boundary values coincide with $\psi(x)$ and which satisfy the inequality $z(x, y) \leqslant R$, and by $W^{+}_{\psi,R}$ the set of those functions in $\overline{W}^{+}_{\psi,R}$ which satisfy the inequality $z(x, y) < R$. The set $\overline{W}^{+}_{\psi,R}$ is nonempty if and only if $\max\limits_{\Gamma} \psi(X) \leqslant R$; analogously, the inequality $\max\limits_{\Gamma} \psi(x) < R$ is the condition for nonemptiness of the set $W^{+}_{\psi,R}$. The set W^{+}_{ψ,R_0} is obviously nonempty. Let $z \in W^{+}_{\psi,R_0}$. Then everywhere in $\Omega + \Gamma$, $z(x, y) < R$. Hence it follows that $\|z\|_C < R_0$; thus $z \in W^{+}_{\infty,\|z\|_C}$ and $\|z\|_C < R_0$. Put $\varphi_z(x, y) = f(x, y, z, p, q)$, where $z = z(x, y)$, and p, q are the coefficients of the supporting plane to the surface $z = z(x, y)$ at the point $(x, y, z(x, y))$. Since

$$0 \leqslant \min_{\Gamma} \psi(x) \leqslant \min_{\Omega} z(x, y) \leqslant \max_{\Omega} z(x, y) < R_0,$$

the function $\varphi_z(x, y)$ is defined in Ω almost everywhere uniquely. Since

$$0 \leqslant \varphi_z(x, y) \leqslant f_{\|z\|_C}(x, y),$$

we have

$$\iint_{\Omega} \varphi_z(x, y)\,dx\,dy \leqslant F(\|z\|_C) < M(+\infty).$$

From property b) of the function f it follows that the function $\varphi_z(x, y)$ satisfies all the conditions of Theorem 3. Thus, for every function $\varphi_z(x, y)$ the function $A_R\varphi_z$ is defined, i.e., on W^{+}_{ψ,R_0} the operator $Bz = A_R\varphi_z(x, y)$ is defined.

Theorem 6. The operator B is completely continuous on each set $\overline{W}^{+}_{\psi,R}$, $R \in [0, R_0)$, in the sense of uniform convergence.

Theorem 7. If there exists a number $\widetilde{R}$ satisfying the inequalities

$$\max_{\Gamma} \psi(x) \leqslant \widetilde{R} < R_0; \qquad F(\widetilde{R}) \leqslant M\left(\widetilde{R} - \max_{\Gamma} \psi(X)\right), \tag{9}$$

then the boundary-value problem

$$rt - s^2 = R(p,q) f(x,y,z,p,q), \qquad z|_{\Gamma} = \psi(X)$$

has a solution in $\overline{W}^{+}_{\psi,\widetilde{R}}$.

Conditions (9) make it possible to apply to the operator equation $z = Bz$ on the set $\overline{W}^{+}_{\psi\widetilde{R}}$ the well-known Schauder fixed-point principle for completely continuous transformations of convex sets into themselves in Banach spaces, whence Theorem 7 follows.

An analogous method can be applied to the study of the Dirichlet problem for strongly elliptic Monge—Ampère equations

$$rt - s^2 = A(x,y,z,p,q)r + 2B(x,y,z,p,q)s + C(x,y,z,p,q)t +$$

$$+\varphi(x,y,z,p,q).$$

In conclusion, we note that Theorem 7 contains, as special cases, the theorems of I. Ya. Bakelman and A. V. Pogorelov cited at the beginning of the article.

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