ON THE FIRST BOUNDARY-VALUE PROBLEM FOR SOME SEMI-BOUNDED HYPOELLIPTIC DIFFERENTIAL OPERATORS

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Abstract

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ON THE FIRST BOUNDARY-VALUE PROBLEM FOR SOME SEMI-BOUNDED HYPOELLIPTIC DIFFERENTIAL OPERATORS

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In this note the first boundary-value problem is studied for certain classes of hypoelliptic equations, as well as for certain classes of equations with semi-bounded principal part.

I. Consider, in a bounded domain Q of n-dimensional space $x = (x_1, \dots, x_n)$, the differential equation of order $2m_1$ in x_1, $2m_2$ in $x_2, \dots, 2m_n$ in x_n, for certain integers $m_1, \dots, m_n,\ m_i > 0$:

$$L(u) \equiv \sum_{|\alpha|=1} A^{(\alpha)}(x) D^\alpha u + \sum_{|\alpha|<1} A^{(\alpha)}(x) D^\alpha u \equiv L_0(u) + L_1(u) = f, \qquad (1)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $|\alpha| = \dfrac{\alpha_1}{2m_1} + \cdots + \dfrac{\alpha_n}{2m_n}$; the functions $A^\alpha(x)$ have in $\overline{Q}$ bounded derivatives up to order

$$\left[\frac{\alpha_1 + \cdots + \alpha_n}{2}\right].$$

Suppose that

$$\inf_{x \in \overline{Q}} \operatorname{Re} \sum_{|\alpha|=1} A^{(\alpha)}(x)(i\xi)^\alpha \geq \theta^2 \left(\xi_1^{2m_1} + \cdots + \xi_n^{2m_n}\right), \quad \theta > 0, \qquad (2)$$

where $(i\xi)^\alpha = (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n}$.

Lemma 1. Equation (1) is hypoelliptic in $\overline{Q}$ (^{1}).

We shall be interested in the question of finding, in the domain Q, a solution of equation (1) satisfying in the usual sense the boundary conditions

$$u|_\Gamma = \cdots = D_x^{m-1} u|_\Gamma = 0, \qquad (3)$$

where $m = \max_i(m_i)$, and Γ is the boundary of the domain Q.

Denote by $\mathring{W}^{(r)}(Q)$, $r = (r_1, \dots, r_n)$, the Sobolev space $\mathring{W}^{(r)}_{x,2}(Q)$ obtained by completing $C_0^\infty(\overline{Q})$ in the metric $W^{(r)}_{x,2}(Q)$; by $\mathring{W}^{(-r)}(Q)$, as usual, we denote the space of functionals on $\mathring{W}^{(r)}(Q)$.

By a generalized solution of problem (1), (3) we shall mean a function

$$u \in \mathring{W}^{(m)}(Q), \quad m = (m_1, \dots, m_n),$$

which, for every function $v \in \mathring{W}^{(m)}(Q)$, satisfies the integral identity

$$(R_1(u), R_2(v)) + (T_1(u), T_2(v)) = (f, v), \tag{4}$$

where the brackets (,) denote the scalar product in $\mathcal{L}_2(Q)$; R_i and T_i, $i = 1, 2$, are operators bounded in $\mathring{W}^{(m)}(Q)$, and for $v \in \mathring{W}^{(m)}(Q)$ and $u \in \mathring{W}^{(m)}(Q) \cap W^{(2m)}(Q)$

$$(R_1 u, R_2 v) = (L_0 u, v), \qquad (T_1 u, T_2 v) = (L_1 u, v)$$

(the possibility of such a decomposition of L_0 and L_1 into R_1, R_2 and T_1, T_2, respectively, is proved).

Lemma 2. There exists a constant λ_0 such that, for any function $u \in C_0^\infty(\overline{Q})$, the inequality

$$(L_0 u, u) \geq \frac{\theta^2}{2} \|u\|^2_{\dot{W}^{(m)}(Q)} - \lambda_0 \|u\|^2_{L_2(Q)}, \tag{5}$$

holds, where θ is the constant from (2).

This lemma is analogous to Gårding' s lemma (^{2}) for elliptic equations. It shows that the bounded operator $R_2^* R_1 + \lambda$, acting in $\dot{W}^{(m)}(Q)$, has a bounded inverse for $\lambda \geq \lambda_0$, which in turn makes it possible to prove the following theorem:

Theorem 1. Problem (1), (3), for $f \in \dot{W}^{(-m)}(Q)$, is regularly solvable in $\dot{W}^{(m)}(Q)$.

Moreover, if the homogeneous problem ($f = 0$) has only the zero solution, then the operator L establishes a one-to-one correspondence between $\dot{W}^{(m)}(Q)$ and $\dot{W}^{(-m)}(Q)$ (by regular solvability, following Vishik, we mean Fredholmness).

II. We now consider, in a bounded domain Q of the $(n+1)$-dimensional space $x = (x_0, x_1, \dots, x_n)$, an equation of odd order $2m_0 + 1$ with respect to x_0 and of even orders $2m_i$ with respect to x_i, $i = 1, \dots, n$, where $m_0 \geq 0$, $m_i > 0$, $i = 1, \dots, n$, $2m_0 + 1 < 2m = \max_{1 \leq i \leq n} 2m_i$,

$$L(u) \equiv \sum_{|\alpha|=1} A^{(\alpha)}(x) D^{\alpha} u + \sum_{\alpha \in E_P} A^{\alpha}(x) D^{\alpha} u \equiv L_0(u) + L_1(u) = f, \tag{6}$$

where $\alpha = (\alpha_0, \dots, \alpha_n)$,

$$|\alpha| = \frac{\alpha_0}{2m_0 + 1} + \frac{\alpha_1}{2m_1} + \dots + \frac{\alpha_n}{2m_n},$$

and $A^{(\alpha)}(x)$ have, in $\overline{Q}$, bounded derivatives up to order

$$\left[\frac{\alpha_0 + \alpha_1 + \dots + \alpha_n + 1}{2}\right].$$

We shall assume that

$$\inf_{x \in \overline{Q}} \left|A^{(2m_0+1,0,0,\dots,0)}(x)\right| \neq 0; \tag{7}$$

$$\inf_{x \in \overline{Q}} \operatorname{Re} \sum_{|\alpha|=1} A^{(\alpha)}(x)(i\xi)^{\alpha} \equiv \inf_{x \in \overline{Q}} P(\xi i) \geq \theta^2 \left(|\xi|^{f_1} + \dots + |\xi|^{f_N}\right) \geq \theta^2 \left(\xi_1^{2m_1} + \dots\right.$$

$$\left.\dots + \xi_n^{2m_n}\right), \qquad \theta > 0. \tag{8}$$

We note that for parabolic equations, which are a special case of (6), in ([3]) an even stronger solution of problem (6), (3) has been constructed.

Denote by $F_P(x)$ the set of those α of the plane $|\alpha| = 1$ for which $A^{\alpha}(x) \neq 0$ enters into $P(\xi i)$. The minimal convex polyhedron in the plane $|\alpha| = 1$ containing all $F_P(x)$ will be called G_P (we note that, by virtue of inequality (8), the vertices $f_1, \dots, f_N$ are vectors with even coordinates). Finally, by M_P we denote the set of points of the $(n+1)$-dimensional simplex $\alpha_i \geq 0$, $i = 0, 1, \dots, n$, $|\alpha| \leq 1$, whose projection onto the plane $\alpha_i = 0$ coincides with the projection onto the same plane of the set G_P, for $i = 1, 2, \dots, n$.

Now one can also define the set E_P, appearing in the definition (6) of the operator $L_1(u)$ of lower-order terms: E_P is the totality of integer points of the set

$$I_P = \left(\alpha_i \geq 0,\ i = 0, 1, \dots, n,\ \frac{\alpha_0}{2m_0} + \frac{\alpha_1}{2m_1} + \dots + \frac{\alpha_n}{2m} < 1\right) \cup$$

$$\cup \left[M_P \cap \left(\frac{\alpha_0}{2m_0 + 1} + \frac{\alpha_1}{2m_1} + \dots + \frac{\alpha_n}{2m_n} < 1 \leq \frac{\alpha_0}{2m_0} + \frac{\alpha_1}{2m_1} + \dots + \frac{\alpha_n}{2m_n}\right)\right].$$

Lemma 3. Equation (6) is hypoelliptic in $\overline{Q}$.

Denote by $H_0(Q)$ and $H_1(Q)$ the Hilbert spaces of functions obtained by completing $C_0^\infty(\overline{Q})$ in the metrics

$$(u,u)_{H_0(Q)} = \left\|D_0^{m_0}u\right\|^2_{L_2(Q)} + (P(D)u,u),$$

$$(u,u)_{H_1(Q)} = \left\|D_0^{m_0+1}u\right\|^2_{L_2(Q)} + (P(D)u,u),$$

respectively. In view of inequality (8), the following embeddings are evident:

$$\overset{\circ}{W}{}^{(m_0,m_1,\dots,m_n)}(Q) \supset H_0(Q); \qquad \overset{\circ}{W}{}^{(m_0+1,m_1,\dots,m_n)}(Q) \supset H_1(Q).$$

By a generalized solution of problem (6), (3) we shall mean a function $u \in H_0(Q)$ for which the equality

$$(R_0u, R_1v) + (T_0u, T_1v) = (f, v) \tag{9}$$

holds for every $v \in H_1(Q)$. The operators R_0, T_0 and R_1, T_1 are bounded, respectively, in the spaces $H_0(Q)$ and $H_1(Q)$; moreover, for $u \in C_0^\infty(\overline{Q})$ and $v \in H_1(Q)$, $(R_0u, R_1v) = (L_0u, v)$, $(T_0u, T_1v) = (L_1u, v)$ (the possibility of such a splitting of the operators L_0 and L_1 is proved).

Assume, for simplicity, that the boundary Γ of the domain Q has only two points $A = (x_0^a, \dots, x_n^a)$ and $B = (x_0^b, x_1^b, \dots, x_n^b)$ at which the tangent plane to Γ is orthogonal to the axis Ox_0 (let $x_0^b > x_0^a$).

Theorem 2. Problem (6), (3), for $f \in \overline{H}_1(Q)$ ($\overline{H}_1(Q)$ is the space conjugate to the space $H_1(Q)$), is regularly solvable in $H_0(Q)$, if Q lies in one of the cylinders

$$|x_{i_k} - x_{i_k}^a| = \psi(x_0 - x_0^a), \qquad x_0 \geq x_0^a,$$

and in one of the cylinders

$$|x_{i_k} - x_{i_k}^b| = \psi(x_0^b - x_0), \qquad x_0 \leq x_0^b,$$

where i_k are m of the numbers $i = 1, 2, \dots, n$ for which $2m_i = 2m$, $\psi(t) = t^{\frac{2m_0+1}{2m}+\gamma}$, and $\gamma > 0$ is some constant*.

Moreover, if the homogeneous problem ($f = 0$) has only the trivial solution, then the operator L establishes a one-to-one correspondence between $H_0(Q)$ and $\overline{H}_1(Q)$.

For the proof of Theorem 2 it is enough to establish unique solvability in $H_0(Q)$ of equation (6) in which $L_1(u) = \lambda u$ for a sufficiently large constant λ (equation (9) with $(T_1 u, T_2 v) = \lambda(u, v)$ will be called equation $(9')$).

The proof of the existence of at least one solution of $(9')$ is carried out according to the same plan as the corresponding proof in $(^3)$, using the following lemma.

Lemma 4. There exists a constant $\lambda_0 > 0$ such that

$$(L_0 u, u) \geq \frac{\theta^2}{2} \|u\|^2_{H_0(Q)} - \lambda_0 \|u\|^2_{L_2(Q)}$$

for all $u \in C_0^\infty(\overline{Q})$.

The proof of uniqueness of the solution constructed is carried out as follows. First it is proved that, for a solution $u \in H_0(Q)$ $(f = 0)$, when $\sigma > 0$ the equality

$$(\widetilde{R}_0 u, \widetilde{R}_1 u[\varphi(x_0)]^\sigma) + \sigma(\widetilde{R}_0 u, \widetilde{R}_1 u[\varphi(x_0)]^{\sigma-1}\varphi'(x_0)) +$$

$$+ \sigma B(u, \sigma) + \lambda(u, u[\varphi(x_0)]^\sigma) = 0, \tag{10}$$

holds.

* For the parabolic equation (6) of order $2m$ $(m_0 = 0,\ m_i = m,\ i = 1, \dots, n)$, unique solvability of the problem under consideration has been proved under the condition that

$$\int_0 \frac{dt}{|\psi(t)|^{2m}} = \infty.$$

where $\varphi(x_0)$ is an infinitely differentiable function on the interval $[x_0^a, x_0^b]$ which, for $x_0^a \leq x_0 \leq x_0^1$, is equal to $x_0 - x_0^a$, for $x_0^2 \leq x_0 \leq x_0^b$ is equal to $x_0^b - x_0$, and on the interval $x_0^1 \leq x_0 \leq x_0^2$ (x_0^1 and x_0^2 are arbitrary points of (x_0^a, x_0^b)) $\varphi(x_0)$ is an arbitrary smooth function with $\varphi(x_0) \geq a > 0$; $\widetilde{R}_0, \widetilde{R}_1$ and $\widetilde{\widetilde{R}}_0, \widetilde{\widetilde{R}}_1$ are operators bounded in $H_0(Q)$, which are parts of the operators R_0 and R_1 ($\widetilde{R}_0$ and $\widetilde{R}_1$ are those parts of R_0 and R_1 which correspond to even $a_0 + a_1 + \dots + a_n$ in $L_0(u)$ (6), while $\widetilde{\widetilde{R}}_0$ and $\widetilde{\widetilde{R}}_1$ are analogously connected with odd $a_0 + a_1 + \dots + a_n$); $B(u, \sigma)$ is a quadratic form with respect to the function u and its derivatives; the coefficients in $B(u, \sigma)$ depend on σ and are bounded for $0 \leq \sigma \leq \sigma_0$, for any fixed σ_0. It is proved that

$$|B(u, \sigma)| + \left|(\widetilde{\widetilde{R}}_0 u, \widetilde{\widetilde{R}}_1 u[\varphi(x_0)]^{\sigma-1}\varphi'(x_0))\right| \leq \tag{11}$$

$$\leq \lambda_1 \left(u, u[\varphi(x_0)]^\sigma\right) + C \min_{\overline{Q}} \left|A^{(2m_0+1,0)}(x)\right| \left|(D_0^{m_0} u, D_0^{m_0} u[\varphi(x_0)]^{\sigma-1} \varphi'(x))\right|$$

with certain C and λ_1 independent of σ.

It turns out that the solution u for $f = 0$ (or even for $f \in \mathcal{L}_2(Q)$) in fact possesses greater smoothness than that ensured by membership in $H_0(Q)$.

Lemma 5. *The solution from $H_0(Q)$ of problem (6), (3) for $f \in \mathcal{L}_2(Q)$ satisfies the inequality*

$$\left\|D_0^{m_0} u \, |x_0 - x_0^a|^{-1/2}\right\|_{\mathcal{L}_2(Q)} + \left\|D_0^{m_0} u \, |x_0 - x_0^b|^{-1/2}\right\|_{\mathcal{L}_2(Q)} < \infty.$$

Lemma 5 and inequality (11) make it possible to pass to the limit in (10) as $\sigma \to 0$ and, by virtue of Lemma 4, for $\lambda \geq \lambda_0 + \lambda_1$, to conclude that $u = 0$.

Remark. The considerations and results of part I can be extended to certain non-hypoelliptic equations

$$L(u) \equiv L_0(u) + L_1(u) = f,$$

for which

$$\operatorname{Re} L_0(x, i\xi) + \lambda_0 \equiv \mathcal{P}(x, \xi) > 0 \quad \text{for } |\xi| \geq 0, \ x \in \overline{Q}$$

with a certain constant λ_0. The operator of lower-order terms $L_1(u)$ and the skew-symmetric principal part in $L_0(u)$ are assumed, in the natural way, to be subordinate to the polynomial $\mathcal{P}(x, D)$. In this case the solution is sought in the space $H_2(Q)$, obtained by completing $C_0^\infty(\overline{Q})$ in the metric $(\mathcal{P}(x, D)u, u) = \|u\|^2_{H_2(Q)}$. The corresponding considerations can also be carried out for the case analogous to part II.

We note that, for the case analogous to I, some results have been obtained by the variational method in ([4],[5]).

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