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Abstract

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ON THE CONVERGENCE OF THE COLLOCATION METHOD

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1. In the paper ([1]), L. V. Kantorovich proposed a method for the approximate solution of boundary-value problems, which he called interpolational. Later the method was again proposed ([2]) (for the case of ordinary differential equations) under the name of the method of placement. Subsequently it came to be called in the literature ([3]) the collocation (coincidence) method, which in essence corresponds to its content.

The author obtained ([4]) results on the convergence of the collocation method as applied to the solution of the simplest boundary-value problem for ordinary differential equations of even order under polynomial approximation, and also showed the possibility of divergence of the method when equally spaced nodes are chosen as the collocation points. The convergence of the collocation method for integral equations was investigated in ([5,6]). In the paper ([7]), the application of the Galerkin, least-squares, and collocation methods to the solution of a boundary-value problem for ordinary integro-differential equations was considered. However, the possibility of implementing the methods and their convergence are not investigated.

In the present note we formulate some theorems on the convergence of the collocation method for ordinary integro-differential equations, as well as for some integro-differential equations in partial derivatives. Our investigation is based on the general theory of approximate methods developed by L. V. Kantorovich ([5,8]).

2. Consider the ordinary integro-differential equation with constant limits of integration

$$(Kx)(t) \equiv x^{(m)}(t) - \lambda \left[\sum_{s=1}^{m} p_s(t) x^{(m-s)}(t) + \int_a^b \sum_{s=0}^{m} q_s(t,\tau) x^{(m-s)}(\tau)\, d\tau \right] = y(t) \tag{1}$$

under the condition

$$L_{jx} \equiv \sum_{i=0}^{m-1} \left[\sum_{k=1}^{l} a_{ij}^{k} x^{(i)}(\tau_k) + \int_a^b b_{ij}(t) x^{(i)}(t)\, dt \right] = 0 \quad (j = 1, 2, \dots, m), \tag{2}$$

where $a \leq \tau_1 < \tau_2 < \cdots < \tau_l \leq b$, and the coefficients a_{ij}^k and the functions $b_{ij}(t)$, summable on $[a, b]$, are such that $\lambda = 0$ is not an eigenvalue of the corresponding homogeneous problem. This assumption makes it possible to construct a sequence of polynomials $R_k(t)$ of degree $(m + k - 1)$ such that $L_{iR}k = 0$ $(k = 1, 2, \dots;\ i = 1, 2, \dots, m)$.

According to the collocation method, we seek an approximate solution of problem (1), (2) in the form

$$x_n(t) = \sum_{k=1}^{n} c_{kR} k(t) = \sum_{k=1}^{m+n} d_{kt}^{k-1}, \tag{3}$$

and determine the coefficients $c_1, c_2, \dots, c_n$ from the system of linear algebraic-differential equations

$$(Kx_n)(t_i) = y(t_i) \qquad (i = 1, 2, \dots, n), \tag{4}$$

where $t_1, t_2, \dots, t_n$ is a prescribed system of nodes in $[a, b]$.

We introduce the conditions:

A. For some r, on $[a, b]$ the following inclusions hold: $y^{(r)}(t) \in \operatorname{Lip}\alpha$, $p_s^{(r)}(t) \in \operatorname{Lip}\alpha$, $\alpha > 0$ $(s = 1, 2, \dots, m)$.

B. For $a \leq t, \tau \leq b$, the $q_s(t, \tau)$ have continuous partial derivatives with respect to the argument t up to order r inclusive, and $\partial^r q_s / \partial t^r \in \operatorname{Lip}\alpha$ with respect to t, uniformly with respect to τ $(s = 0, 1, \dots, m)$.

C. λ is not an eigenvalue of the problem.

Theorem 1. *Suppose that the Chebyshev nodes or the Gauss nodes are chosen as the collocation points. If, for $r \geq 0$, conditions A, B, C* are fulfilled, then for all sufficiently large n the system (4) is solvable and the approximate solutions x_n^* converge uniformly, together with their derivatives up to order m inclusive, to the exact solution x^* of the problem (1), (2) and to its corresponding derivatives, with rate*

$$\max_{a \leq t \leq b} \left| x^{*(k)}(t) - x_n^{*(k)}(t) \right| = O\left(\frac{\ln n}{n^{r+\alpha}} \right) \tag{5}$$

for the Chebyshev nodes, and

$$\max_{a\le t\le b}\left|x^{*(k)}(t)-x_n^{*(k)}(t)\right| = O\left(\frac{1}{n^{r+\alpha-\frac{1}{2}}}\right) \tag{6}$$

for the Gauss nodes $(k=0,1,\dots,m)$.

Theorem 2. *Suppose that the roots of the* n*-th orthogonal polynomial with weight* $\omega(t)\ge\gamma>0$*, bounded below, are chosen as the collocation points. If* $p_1=0$*** *and, for* $r\ge 1$*, conditions A, B, C are fulfilled, then for all sufficiently large* n *the system (4) is solvable and* x_n^* *converges to* x^* *with rate*

$$\max_{a\le t\le b}\left|x^{*(k)}(t)-x_n^{*(k)}(t)\right| = O\left(\frac{1}{n^{r+\alpha-1}}\right)\qquad (k=0,1,\dots,m). \tag{7}$$

3. Consider problem (2) for the integro-differential equation with variable limits of integration

$$(K_1x)(t)\equiv x^{(m)}(t)-\lambda\left[\sum_{s=1}^{m}p_s(t)x^{(m-s)}(t)+\int_a^t\sum_{s=0}^{m}q_s(t,\tau)x^{(m-s)}(\tau)\,d\tau\right]=y(t). \tag{8}$$

We still seek the solution of problem (8), (2) in the form (3), and determine the coefficients $c_1,c_2,\dots,c_n$ from the system

$$(K_1x_n)(t_i)=y(t_i), \tag{9}$$

where $a\le t_i\le b$ $(i=1,2,\dots,n)$.

Replace condition B by the following:

B′. For $a\le\tau\le t\le b$, the $q_s(t,\tau)$ have continuous partial derivatives with respect to t up to order r inclusive, and $\partial^r q_s/\partial t^r\in\operatorname{Lip}\alpha$ with respect to t, uniformly with respect to τ, and $d^{r-1}q_s(t,t)/dt^{r-1}\in\operatorname{Lip}\alpha$ $(s=0,1,\dots,m)$.

Theorem 3. *Suppose that the Chebyshev nodes or the Gauss nodes are chosen as the collocation points. If, for* $r\ge 0$*, conditions A, B′, C*** *are fulfilled, then for all sufficiently large* n *the system (9) is solvable, and the approximate solutions* x_n^* *converge to the exact solution* x^* *of the problem (8), (2) with the rate determined by equalities (5) and (6).*

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[*] For the Gauss nodes, when $r=0$, one must additionally assume $\alpha>1/2$.

[**] This can be achieved by a suitable change of variables.

Theorem 4. *Let the roots of the n-th orthogonal polynomial with a weight bounded below be chosen as the collocation points. If $p_1 = q_0 \equiv 0$ and, for $r \geqslant 1$, conditions A, B′, C are satisfied, then for all sufficiently large n the system (9) is solvable and x_n^* converges to x^* at the rate determined by the inequality (7).*

4. Let $P(x, y)$ and $Q(\xi, \eta)$ be points of the square S $(0 < x, y < \pi)$. Consider the Dirichlet problem for a partial integro-differential equation of the form

$$(K_2 u)(P) \equiv \Delta u(P) - \lambda \left[p(P)u(P) + \iint_S q(P, Q)u(Q)\, d\xi\, d\eta \right] = v(P) \qquad (10)$$

under the condition

$$u(P)\big|_\Gamma = 0, \qquad (11)$$

where Γ is the contour of S.

We seek an approximate solution of problem (10), (11) in the form

$$u_{mn}(x, y) = \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} \sin kx \cdot \sin ly,$$

and the coefficients a_{kl}, according to the collocation method, are determined from the system of linear algebraic equations

$$(K_2 u_{mn})(x_i, y_j) = v(x_i, y_j), \quad (x_i, y_j) \in S \qquad (12)$$

$$(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n).$$

Theorem 5. *Suppose the following conditions are satisfied:*

1) *For $P, Q \in \overline{S}$, the functions $v(P)$, $p(P)$, and $q(P, Q)$ are continuous and satisfy a Lipschitz condition with exponent $\alpha > 0$ in the argument x and with exponent $\beta > 0$ in the argument y (for q this condition is assumed to hold uniformly with respect to Q).*
2) *$v(P)$ and $q(P, Q)$ satisfy condition (11), $Q \in \overline{S}$.*
3) *λ is not an eigenvalue of problem (10), (11).*
4)

$$x_i = \pi \frac{2i - 1}{2m + 1}, \qquad y_j = \pi \frac{2j - 1}{2n + 1} \quad (i = 1, 2, \dots, m;\ j = 1, 2, \dots, n).$$

Then the system (12) is solvable for all sufficiently large m and n, and the approximate solutions u^*_{mn} converge to the exact solution u^* of problem (10), (11) at the rate

$$\max_{P\in\overline{S}} |\Delta\left[u^*(P) - u^*_{mn}(P)\right]| = O\left[\ln^2 m \cdot \ln^2 n \cdot \left(\frac{1}{m^\alpha} + \frac{1}{n^\beta}\right)\right].$$

5. Consider the partial integro-differential equation of the form

$$(K_3 u)(P,t) \equiv \left(\Delta - \frac{\partial}{\partial t}\right) u(P,t) - \lambda\left[p(P,t)u(P,t) + \int_0^T \iint_S q(P,t;Q,\tau)u(Q,\tau)\, d\xi\, d\eta\, d\tau\right] = v(P,t) \tag{13}$$

with homogeneous initial and boundary conditions

$$u(P,0) = 0, \quad P \in \overline{S}; \qquad u(P,t)\big|_\Gamma = 0, \quad 0 \leqslant t \leqslant T. \tag{14}$$

We seek an approximate solution of problem (13), (14) in the form

$$u_{mn}(x,y,t) = \sum_{k=1}^{m}\sum_{l=1}^{n} f_{kl}(t) \sin kx \cdot \sin ly, \tag{15}$$

and the functions $f_{kl}(t)$ are determined from the condition that equation (13) be satisfied on the lines $x = x_i,\ y = y_j,\ 0 < x_i,\ y_j < \pi\ (i = 1,2,\dots,m;\ j = 1,2,\dots,n)$, i.e., from the system of ordinary integro-differential equations of the first order with constant limits of integration

$$(K_3 u_{mn})(x_i, y_j; t) = v(x_i, y_j, t) \quad (i = 1,2,\dots,m;\ j = 1,2,\dots,n) \tag{16}$$

under the initial conditions

$$f_{kl}(0) = 0 \quad (k = 1,2,\dots,m;\ l = 1,2,\dots,n). \tag{17}$$

With this choice of the functions $f_{kl}(t)$, expression (15) satisfies conditions (14).

Theorem 6. *Suppose that the following conditions are satisfied:*

1) *For $P,Q \in \bar{S}$ and $0 \le t,\tau \le T$, the functions $v(P,t)$, $p(P,t)$, and $q(P,t;Q,\tau)$ are continuous and satisfy a Lipschitz condition with exponent $\alpha > 0$ in the argument x and exponent $\beta > 0$ in the argument y, uniformly with respect to t (for q, moreover, uniformly with respect to Q and τ).*
2) *$v(P,t)$ and $q(P,t;Q,\tau)$ vanish for $P \in \Gamma,\ Q \in \bar{S},\ 0 \le t,\tau \le T$.*

3) λ *is not an eigenvalue of problem (13), (14).*

4) $x_i = \pi \dfrac{2i-1}{2m+1}, \quad y_j = \pi \dfrac{2j-1}{2n+1}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

Then the system (16) with initial conditions (17) is solvable for all sufficiently large m *and* n*, and the approximate solutions* u^*_{mn} *converge to the exact solution* u^* *of problem (13), (14) with the rate*

$$\max_{\substack{P\in\bar{S}\\ 0\le t\le T}} \left|\left(\Delta - \frac{\partial}{\partial t}\right)[u^*(P,t) - u^*_{mn}(P,t)]\right| = O\left[\ln^2 m \cdot \ln^2 n \cdot \left(\frac{1}{m^\alpha} + \frac{1}{n^\beta}\right)\right].$$

Remark. All the theorems presented may be regarded as convergence theorems for certain interpolation processes, when the interpolation polynomials are constructed from the values of integro-differential operators at the nodes and from the boundary (and initial) conditions.

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Note: Figure translations are in progress. See original paper for figures.