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## Abstract

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### THEORY OF ELASTICITY

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# ON THE STABILITY OF A COMPRESSED PLATE WITH ACCOUNT TAKEN OF ROTATION ANGLES

*(Presented by Academician A. Yu. Ishlinskii on January 21, 1962)*

V. V. Novozhilov showed ($^1$) how the influence of rotation angles may be taken into account in the loss of stability of elastic (and not only elastic) bodies. In the works of L. S. Leibenzon ($^2$), A. Yu. Ishlinskii ($^3$), and others, the stability of elastic bodies was investigated from the point of view of the general mathematical theory of elasticity; the influence of rotation angles on the critical loads was not taken into account; in the limiting case, from the solutions found, the well-known formulas of the theory of stability of elastic rods and plates may be obtained.

In the present work, the loss of stability of a thick freely supported rectangular plate, uniformly compressed in one direction, is investigated on the basis of the equations of V. V. Novozhilov' s theory of stability. It is shown that, in the limit for small plate thickness, the formulas of the theory of stability of plates under the Kirchhoff-Love hypotheses ($^4$) do not follow from the obtained solution. The stability of thin plates under the Kirchhoff-Love hypotheses follows only in the case where the influence of rotation angles is neglected ($^6$). This circumstance indicates that the widely used theory of stability of plates (and shells) may be revised from the standpoint of taking into account the influence of rotation angles.

Consider a rectangular freely supported thick plate of thickness $2a$, length $2l$, and width $2b$, compressed along the $z$-axis by forces of intensity $p$ (Fig. 1).

We shall regard the material as incompressible and take as the starting point the known relations of the theory of small elastic-plastic deformations ($^5$)

$$\sigma_x - \sigma = \frac{2}{3}\frac{\sigma_i}{e_i}e_x; \qquad \tau_{xy} = \frac{1}{3}\frac{\sigma_i}{e_i}e_{xy} \quad (x, y, z);$$

$$\sigma_i = \Phi(e_i); \qquad e_x + e_y + e_z = 0;$$

$$\sigma_i = \frac{\sqrt{2}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)}; \tag{1}$$

$$e_i = \frac{\sqrt{2}}{3}\sqrt{(e_x - e_y)^2 + (e_y - e_z)^2 + (e_z - e_x)^2 + \frac{3}{2}(e_{xy}^2 + e_{xz}^2 + e_{yz}^2)},$$

where $(x, y, z)$ denotes a cyclic permutation of the indices.

We seek the solution of the problem in the form

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}'; \qquad e_{ij} = e_{ij}^0 + e_{ij}'; \qquad u = u^0 + u'; \dots, \tag{2}$$

where the superscript zero denotes the state of equilibrium before loss of stability, and the prime superscript denotes increments of the components during loss of stability.

We take the equilibrium conditions in the form ($^1$)

$$\frac{\partial}{\partial x}\left[\sigma_x' - \omega_z'\tau_{xy}^0 + \omega_y'\tau_{xz}^0\right] + \frac{\partial}{\partial y}\left[\tau_{xy}' - \omega_z'\sigma_y^0 + \omega_y'\tau_{yz}^0\right] +$$

$$+\frac{\partial}{\partial z}\left[\tau_{xz}' - \omega_z'\tau_{yz}^0 + \omega_y'\sigma_z^0\right] = 0 \quad (x, y, z), \tag{3}$$

where

$$\omega_x' = \frac{1}{2}\left(\frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z}\right); \qquad \omega_y' = \frac{1}{2}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right); \qquad \omega_z' = \frac{1}{2}\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right). \tag{4}$$

For the initial equilibrium position at $z = \pm l$ we shall have

$$\sigma_x^0 = \sigma_y^0 = 0; \qquad \sigma_z^0 = -p; \qquad \tau_{xy}^0 = \tau_{xz}^0 = \tau_{yz}^0 = 0; \qquad e_{xy}^0 = e_{xz}^0 = e_{yz}^0 = 0. \tag{5}$$

From the incompressibility condition we obtain

$$e_x^0 = e_y^0 = -\frac{1}{2}e_z^0. \tag{6}$$

Fig. 1

Figure 1: Fig. 1

**Fig. 1**

Linearizing the relations (1), taking into account (2), (5), (6), and the incompressibility of the material, we find

$$\begin{gathered}
\sigma_i' = -p; \qquad \sigma_i' = \sigma_z' - \frac{1}{2}(\sigma_y' + \sigma_x'); \qquad e_i^0 = e_z^0; \qquad e_i' = e_z'; \\
\sigma_x' - \sigma' = -B_0(e_x' - e_y') - \frac{1}{6}(2\sigma_z' - \sigma_y' - \sigma_x'); \\
\sigma_y' - \sigma' = -B_0(e_y' - e_x') - \frac{1}{6}(2\sigma_z' - \sigma_y' - \sigma_x'); \\
\tau_{xy}' = -B_0 e_{xy}' = -B_0 \left( \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) \qquad (x, y, z),
\end{gathered} \tag{7}$$

where $B_0 = p/3e_z^0$.

Taking (5), (6) into account, we write the equilibrium conditions (3) in the form

$$\begin{gathered}
\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + \frac{\partial}{\partial z}\left(\tau_{xz}' - \omega_y' p\right) = 0; \\
\frac{\partial \tau_{yx}'}{\partial x} + \frac{\partial \sigma_y'}{\partial y} + \frac{\partial}{\partial z}\left(\tau_{zy}' + \omega_x' p\right) = 0; \\
\frac{\partial \tau_{zx}'}{\partial x} + \frac{\partial \tau_{zy}'}{\partial y} + \frac{\partial \sigma_z'}{\partial z} = 0.
\end{gathered} \tag{8}$$

Substituting $\tau_{ij}$ from (7) and bearing in mind (4), we find

$$\begin{gathered}
\frac{\partial \sigma_x'}{\partial x} = B_0 \left[ \Delta u' - 2\frac{\partial^2 u'}{\partial x^2} \right] + \frac{p}{2} \left[ \frac{\partial^2 u'}{\partial z^2} - \frac{\partial^2 w'}{\partial x\, \partial z} \right]; \\
\frac{\partial \sigma_y'}{\partial y} = B_0 \left[ \Delta v' - 2\frac{\partial^2 v'}{\partial y^2} \right] + \frac{p}{2} \left[ \frac{\partial^2 v'}{\partial z^2} - \frac{\partial^2 w'}{\partial y\, \partial z} \right]; \\
\frac{\partial \sigma_z'}{\partial z} = B_0 \left[ \Delta w' - 2\frac{\partial^2 w'}{\partial z^2} \right],
\end{gathered} \tag{9}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

From (7) and (9), by comparing analogous expressions for the difference $\sigma_x' - \sigma_y'$, we obtain

$$\frac{\partial \Delta u'}{\partial y} - \frac{\partial \Delta v'}{\partial x} + \frac{q}{2} \left[ \frac{\partial^3 u'}{\partial y\, \partial z^2} - \frac{\partial^3 v'}{\partial x\, \partial z^2} \right] = 0, \tag{10}$$

where $q = p/B_0 = 3e_z^0$.

Expanding $\sigma_i = \Phi(e_i)$ in a series and restricting ourselves to the linear approximation, we find

$$2\sigma_z' - \sigma_y' - \sigma_x' = A_0 e_z', \tag{11}$$

where $A_0 = 2(d\Phi/de_i)_{e_i=e_i^0}$.

Differentiating (11) with respect to $x, y, z$ and comparing with the analogous expression obtained from relations (9), we find

$$2\frac{\partial^3 \Delta u'}{\partial x^2 \partial y} + \frac{\partial^3 \Delta u'}{\partial y\, \partial z^2} + 2\frac{\partial^3 \Delta v'}{\partial x\, \partial y^2} + \frac{\partial^3 \Delta v'}{\partial x\, \partial z^2} - (k + 6 - q)\left[\frac{\partial^5 u'}{\partial x^2 \partial y\, \partial z^2} + \frac{\partial^5 v'}{\partial x\, \partial y^2 \partial z^2}\right] + \frac{q}{2}\left[\frac{\partial^5 u'}{\partial y\, \partial z^4} + \frac{\partial^5 v'}{\partial x\, \partial z^4}\right] = 0, \tag{12}$$

where $k = A_0/B_0$.

In the system of differential equations (10), (12), the terms with coefficient $q$ take into account the influence of rotation. Let us note that the solution without taking the angles of rotation into account can be obtained for $q = 0$.

We seek the solution of the system of equations (10), (12) in the form

$$u' = F(x)\cos(ny)\cos(mz); \qquad v' = G(x)\sin(ny)\cos(mz). \tag{13}$$

Then the unknown functions are written in terms of $F$ and $G$ as follows:

$$\begin{aligned}
w' &= -\frac{1}{m}\,[F' + nG]\cos(ny)\sin(mz);\\
\Delta u' &= [F'' - \beta^2 F]\cos(ny)\cos(mz);\\
\Delta v' &= [G'' - \beta^2 G]\sin(ny)\cos(mz);\\
\tau_{xy}' &= B_0[nF - G']\sin(ny)\cos(mz);\\
\tau_{xz}' &= \frac{B_0}{m}[F'' + m^2 F + nG']\cos(ny)\sin(mz);\\
\tau_{yz}' &= -\frac{B_0}{m}[nF' + (n^2 - m^2)G]\sin(ny)\sin(mz);\\
\sigma_x' &= \frac{B_0}{2}\left[(q-2)F' + nqG - 2\xi^2\int F\,dx\right]\cos(ny)\cos(mz);\\
\sigma_y' &= -\frac{B_0}{n}\left[G'' + (n^2 - m^2)G - \frac{q}{2}(nF' + \beta^2 G)\right]\cos(ny)\cos(mz);\\
\sigma_z' &= -\frac{B_0}{m^2}\left[(n^2 - m^2)(F' + nG) - (F''' + nG'')\right]\cos(ny)\cos(mz),
\end{aligned} \tag{14}$$

where

$$\beta^2 = m^2 + n^2; \qquad 2\xi^2 = 2\beta^2 + qm^2. \tag{15}$$

The prime on $F$ and $G$ denotes derivatives with respect to $x$. The constants of integration have been omitted, which is connected with the satisfaction of the boundary conditions.

Substituting (13) into the system (10), (12), we rewrite it in the form

$$\begin{aligned} &nF'' + G''' - \xi^2(nF + G') = 0; \\ &2n\left[F^{(\mathrm{IV})} - \beta^2 F''\right] - nm^2\left[F'' - \beta^2 F\right] + (2n^2 + m^2)\left[G''' - \beta^2 G'\right] \\ &+ (k + 6 - q)nm^2\left[F'' + nG'\right] + \frac{m^4 q}{2}\left[nF - G'\right] = 0. \end{aligned} \tag{16}$$

Solving this system, we find $F$ and $G$, which will be expressed as sums of even and odd functions.

Considering the case of lateral buckling of the plate, it is necessary to take the function $F$ to be even. Thus, we shall have

$$\begin{aligned} F(x) &= A_1 \operatorname{ch}(\lambda x) + A_2 \operatorname{ch}(\gamma x) + DA_3 \operatorname{ch}(\xi x); \\ G(x) &= -\frac{A_1 n}{\lambda}\operatorname{sh}(\lambda x) - \frac{A_2 n}{\gamma}\operatorname{sh}(\gamma x) + (1 - nD)\frac{A_3}{\xi}\operatorname{sh}(\xi x), \end{aligned} \tag{17}$$

where

$$\begin{aligned} D &= -\frac{2n}{(2+q)m^2}; \\ \lambda &= \frac{1}{2}\sqrt{4n^2 - (2 + k - q)m^2 + m^2\sqrt{(6 + k - q)^2 - 8(6 + k)}}; \\ \gamma &= \frac{1}{2}\sqrt{4n^2 - (2 + k - q)m^2 - m^2\sqrt{(6 + k - q)^2 - 8(6 + k)}}. \end{aligned} \tag{18}$$

For a freely supported plate, $u' = 0$ at $y = \pm b$, $z = \pm l$. Then from (13) and (14) we obtain

$$ml = \pm\pi/2 + i\pi; \qquad nb = \pm\pi/2 + j\pi; \qquad i, j = 0, 1, 2, \ldots; \tag{19}$$

$$\sigma'_y = 0 \text{ at } y = \pm b; \qquad \sigma'_z = 0 \text{ at } z = \pm l.$$

Consequently, when the plate buckles, its faces $y = \pm b$, $z = \pm l$ are free of bending and twisting moments.

The boundary conditions on the lateral surface $x = \pm a$, free of load, are written in the form

$$\begin{aligned} \sigma'_x &= 0; \\ \tau'_{xy} &= 0 \quad \text{for } x = \pm a; \\ \tau'_{xz} + p\,\frac{\partial u'}{\partial z} &= 0, \end{aligned} \tag{20}$$

whence

$$\begin{aligned} (q-2)F' + nqG - 2\xi^2 \int F\,dx &= 0; \\ nF - G' &= 0; \\ F'' + m^2(1 - 3e_z^0)F + nG' &= 0 \end{aligned} \tag{21}$$

for $x = \pm a$.

System (21), after the appropriate substitutions and transformations, gives three equations for the unknown constants $A_1, A_2, A_3$, from which we find the transcendental equation for determining the critical value of the strain and, consequently, of the load, in the form

$$\begin{aligned} e_z^0 = \Bigg\{ &\frac{1}{\lambda}\,[q(\lambda^2-\beta^2) - 2(\lambda^2+\beta^2)][2nD(\xi^2-\gamma^2) + (\gamma^2+\beta^2)]\ \mathrm{th}(\lambda a) \\ &- \frac{1}{\gamma}\,[q(\gamma^2-\beta^2) - 2(\gamma^2+\beta^2)][2nD(\xi^2-\lambda^2) + (\lambda^2+\beta^2)]\ \mathrm{th}(\gamma a) \\ &+ \frac{2n(\gamma^2-\lambda^2)}{\xi}[Dq(\xi^2-\beta^2) + nq - 2D(\xi^2+\beta^2)]\ \mathrm{th}(\xi a) \Bigg\} : \\ &: 3m^2 \left\{ \frac{1}{\lambda}\,[q(\lambda^2-\beta^2) - 2(\lambda^2+\beta^2)]\ \mathrm{th}(\lambda a) - \frac{1}{\gamma}\,[q(\gamma^2-\beta^2) - 2(\gamma^2+\beta^2)]\ \mathrm{th}(\gamma a) \right\}. \end{aligned} \tag{22}$$

Here the quantities $\lambda, \gamma, \xi, D$ depend on $q$. Equation (22) can be solved by successive approximations, taking $q = 0$ in the first approximation.

If the thickness $2a$ is small, then, linearizing (12), we obtain

$$e_z^0 = \Bigg\{\Bigg\{2q[nD(\beta^2-\xi^2)-m^2] + \frac{a^2}{3}\Big\{2nD[\xi^2(\xi^2+\gamma^2+\lambda^2)-\gamma^2\lambda^2] + q[\beta^2(\gamma^2+\lambda^2-\beta^2)+\gamma^2\lambda^2-2n^2\xi^2] + 4nD[\xi^2(\xi^2-\gamma^2-\lambda^2)+\gamma^2\lambda^2] - 2(\lambda^2+\beta^2)(\gamma^2+\beta^2)\Big\}\Bigg\}\Bigg\} / 3(2-q)m^2. \tag{23}$$

Putting $q=0,\ \gamma=i\omega,\ \xi^2=\beta^2,\ D=-n/m^2$, from expressions (22) and (23) we obtain the previously found formulas (6) for determining $e_z^0$, respectively for arbitrary $a$ and for small $a$.

If the loss of stability is not accompanied by passage beyond the yield limit ($k=-6$), then in system (10), (12) one must put $q=0$, and we arrive at the solution given in (6), from which, as a special case, we obtain the formula presented in (4).

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