MATHEMATICS

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Abstract

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TWO THEOREMS ON ERROR ESTIMATION AND SOME OF THEIR APPLICATIONS

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§ 1. Let, in a linear space A, an approximate solution $\tilde{f}$ of the linear equation

$$Kf = g \tag{1}$$

be sought by exactly solving an approximate equation, simpler in structure,

$$\tilde{K}\tilde{f} = \tilde{g}. \tag{2}$$

The following theorems differ from the corresponding results of L. V. Kantorovich (^{1}), Ch. XIV, in particular, in that in these theorems it is not the error $\|f - \tilde{f}\|_A$ that is estimated (we do not introduce a norm in the space A), but the error $\|f - \tilde{f}\|_{A_0}$, where A_0 is a subspace of the space A.

Theorem 1. *Let:* 1) K and $\tilde{K}$ be linear operators, defined in A, whose values lie in the linear space B; 2) equation (1), for the given element $g \in B$, have a solution $f \in A$; 3) $(K - \tilde{K})f_1 \in B_0$, where f_1 is any element of A, and B_0 is a linear normed subspace of the space B; 4) equation (2), where the element $\tilde{g}$ satisfies the condition $g - \tilde{g} \in B_0$, have a unique solution $\tilde{f} \in A$; 5) for every element $\psi \in B_0$ the equation $\tilde{K}\varphi = \psi$ have a solution $\varphi \in A_0$, where A_0 is a linear normed subspace of the space A; 6) $\|(K - \tilde{K})\varphi\|_{B_0} \leq \|K - \tilde{K}\| \cdot \|\varphi\|_{A_0}$ for every element $\varphi \in A_0$; 7) $\|\tilde{K}^{-1}\psi\|_{A_0} \leq \|\tilde{K}^{-1}\| \cdot \|\psi\|_{B_0}$ for every $\psi \in B_0$; 8) $\|K - \tilde{K}\| \cdot \|\tilde{K}^{-1}\| < 1$.*

Then the solution of equation (1) is unique, $f - \tilde{f} \in A_0$, $K\tilde{f} - g \in B_0$, and

$$\|f - \tilde{f}\|_{A_0} \leq \frac{\|\tilde{K}^{-1}\| \cdot \|K\tilde{f} - g\|_{B_0}}{1 - \|\tilde{K}^{-1}\| \cdot \|K - \tilde{K}\|}. \tag{3}$$

Theorem 2. *Let assumptions 4) and 5) of Theorem 1 be replaced by the following:* 4) equation (2), where the element $\tilde{g}$ satisfies the condition $g - \tilde{g} \in B_0$, is solvable in the space A; the homogeneous equation $\tilde{K}\tilde{f}_0 = 0$ has in A one and only one linearly independent solution $\tilde{f}_0$; the general solution of equation (2) belongs to the linear space $A_1 \subseteq A$; $(T, \tilde{f}_0) \neq 0$, where T is a linear functional defined on A_1; 5) for every element $\psi \in B_0$ the equation $\tilde{K}\varphi = \psi$ has a unique solution in the linear normed space $A_0 \subset A_1$; moreover $(T, \xi) = 0$ for every element $\xi \in A_0$.

Then $f \in A_1$*, where* f *is an arbitrarily fixed solution of equation (1). In addition, if* $\tilde{f}$ *is such a solution of equation (2) that* $(T, \tilde{f}) = (T, f)$*, then* $f - \tilde{f} \in A_0$*,* $K\tilde{f} - g \in B_0$*, and estimate (3) holds.*

We note that, in choosing $\tilde{K}$ and $\tilde{g}$, one should strive to make the quantities $\|K - \tilde{K}\|$ and $\|g - \tilde{g}\|_{B_0}$ minimal. We also note that the spac-

the space B_n is not prescribed in advance and thus the solver is given the possibility of a rational choice of this space.

§ 2. Consider the question of an approximate solution of the Wiener-Hopf equation

$$f(x) + \frac{1}{\sqrt{2\pi}} \int_0^\infty k(x-t) f(t)\, dt = g(x), \qquad x > 0. \tag{4}$$

Let $f, g \in L_2(0, \infty)$, $k(x) \in L_2(-\infty, \infty)$, and let the Fourier integral

$$K(x) \equiv \operatorname{l.i.m.}_{\lambda,\lambda_1 \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda_1} k(t) e^{ixt}\, dt$$

satisfy the Hölder condition on the completed x-axis. Suppose, moreover, that $1 + K(x) \neq 0$ and $[\arg(1 + K(x))]_{-\infty}^{\infty} = 0$.

Using Theorem 1, we arrive at the conclusion that, in order to obtain an approximate solution $\tilde{f}(x)$ and an estimate of the mean-square error, it is enough to solve an equation of the form (4), in which, instead of f, k, and g, the functions $\tilde{f}$, $\tilde{k}$, and $\tilde{g}$ stand. Here $\tilde{f} \in L_2(0, \infty)$, $\tilde{g} \in L_2(0, \infty)$, $\tilde{k}(x) \in L_2(-\infty, \infty)$; the Fourier integral $\widetilde{K}(x)$ satisfies the Hölder condition, $1 + \widetilde{K}(x) \neq 0$, $[\arg(1 + \widetilde{K}(x))]_{-\infty}^{\infty} = 0$, and

$$\max |K(x) - \widetilde{K}(x)| \cdot \max |\widetilde{X}^+| \cdot \max |\widetilde{X}^-|^{-1} = \delta < 1.$$

Here $\widetilde{X}^+$, $\widetilde{X}^-$ are the canonical solution of the Riemann problem (see, for example, (^{2}), § 38),

$$\widetilde{X}^+(x)[1 + \widetilde{K}(x)] = \widetilde{X}^-(x), \qquad -\infty < x < \infty.$$

The error estimate has the form

$$\left(\int_0^\infty |f-\tilde f|^2\,dx\right)^{1/2} \le$$

$$\le \frac{\max|\widetilde{X}^+|}{(1-\delta)\min|\widetilde{X}^-|}\left(\int_0^\infty \left|\tilde f(x)+\frac{1}{\sqrt{2\pi}}\int_0^\infty k(x-t)\tilde f(t)\,dt-g(x)\right|^2 dx\right)^{1/2}.$$

As $\tilde k$ and $\tilde g$ one may take, for example, linear combinations of functions of the form $|x|^k\exp(-\alpha|x|+\beta x)$ and $x^k\exp(-\alpha|x|+\beta x)$, whose Fourier integrals are rational functions. The solution $\tilde f$ will also have the form of such a combination. In choosing $\tilde k$ and $\tilde g$, one should seek to reduce to a minimum δ and

$$\int_0^\infty |g-\tilde g|^2\,dx.$$

In (^{3}), for the approximate solution of equation (4) (and of a somewhat more general equation), the method of B. G. Galerkin was used.

§ 3. Consider the equation of the first kind

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty k(x-t)f(t)\,dt = g(x), \qquad x>0, \tag{5}$$

where $g\in L_2(0,\infty)$ and $k(x)\in L_2(-\infty,\infty)$, while the function $K(x)\sqrt{x^2+1}$ satisfies the Hölder condition with exponent not less than 1/2, and does not vanish on the completed axis: $[\arg K(x)\sqrt{x^2+1}]_{-\infty}^{\infty}=0$. The solution is sought in the class of generalized functions f, whose Fourier integrals $F(x)$ are ordinary functions analytically continuable into the upper half-plane, and moreover for all $y\ge 0$

$$\int_{-\infty}^\infty \left|\frac{F(x+iy)}{x+iy+i}\right|^2 dx < \text{const}.$$

Let us note that functions f representable, for small positive values of x, in the form

$$f=\frac{a}{\sqrt{x}}+f_0(x), \qquad a=\text{const}, \qquad f_0\in L_2(0,\varepsilon), \qquad \varepsilon>0 \tag{6}$$

may belong to this class (the notation of equalities (5) and (6) is, of course, conditional, since they contain the generalized function f. This remark also applies to some subsequent formulas).

Using Theorem 2, we arrive at the following result: suppose that equation (5) has a solution f, representable in the form (6), with the number a known. In order to obtain an approximate solution $\tilde{f}$ and an estimate of the mean-square error, it is sufficient to solve, in the indicated class of generalized functions, the equation

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \tilde{k}(x-t)\tilde{f}(t)\,dt = \tilde{g}(x) + Ce^{-qx}, \qquad x > 0, \quad q > 0;$$

the functions $\tilde{k}(x)$ and $\tilde{g}(x)$ and the constants C and q are such that: a) $\tilde{k}(x) \in L_2(-\infty, \infty)$, the function $\widetilde{K}(x)\sqrt{x^2+q^2}$ satisfies a Hölder condition with exponent not less than 1/2, $\widetilde{K}(x)\sqrt{x^2+q^2} \neq 0$, $[\arg \widetilde{K}(x)\sqrt{x^2+q^2}]_{-\infty}^{\infty} = 0$; b) $\max |[K(x) - \widetilde{K}(x)](x^2+q^2)^{3/2}| < \infty$; c)

$$\max |[K(x) - \widetilde{K}(x)](x^2+q^2)| \times \max |\widetilde{X}^+(x^2+q^2)^{-1/4}| \cdot \max |\widetilde{X}^-(x^2+q^2)^{1/4}|^{-1} = \delta < 1;$$

here $\widetilde{X}^+$, $\widetilde{X}^-$ are the canonical solution of the problem

$$\widetilde{X}^+(x)\,\widetilde{K}(x)\sqrt{x^2+q^2} = \widetilde{X}^-(x), \qquad -\infty < x < \infty;$$

d) $\tilde{g}(x) \in L_2(0, \infty)$,

$$\frac{d^2}{dx^2} \left\{ \begin{array}{ll} g(x) - \tilde{g}(x), & x > 0, \\ 0, & x < 0 \end{array} \right\} \in L_2(-\infty, \infty);$$

e) the solution $\tilde{f}$ has, for small positive x, the form (6) with the same constant a. If conditions a)–e) are fulfilled, then $f - \tilde{f} \in L_2(0, \infty)$ and

$$\left(\int_0^\infty |f - \tilde{f}|^2\,dx \right)^{1/2} \leq$$

$$\leq \frac{\max |\widetilde{X}^+(x^2+q^2)^{-1/4}|}{(1-\delta)\min |\widetilde{X}^-(x^2+q^2)^{1/4}|} \left(\int_0^\infty \left| \left(\frac{d^2}{dx^2} - q^2 \right) \left[\frac{1}{\sqrt{2\pi}} \int_0^\infty k(x-t)\tilde{f}\,dt - g(x) \right] \right|^2 dx \right)^{1/2}.$$

Remark 1. If $g(x) = e^{-\alpha x}$, $\alpha > 0$, then

$$a = \sqrt{\frac{\alpha + q}{\pi}} \exp \left(\frac{1}{2\pi i} \int_{-\infty}^\infty [\ln K(t)\sqrt{t^2+q^2}] \frac{dt}{t + i\alpha} \right).$$

Here, without loss of generality, it is assumed that

$$\lim_{x\to\infty} K(x)\sqrt{x^2+q^2} = 1,$$

and $\ln(K(x)\sqrt{x^2+q^2})$ is a continuous function vanishing at infinity.

Remark 2. If, as $\tilde{g}$, one takes the function recommended at the end of § 2, and sets

$$\widetilde{K}(x) = R(x)(x^2+p^2)^{-1/2}, \qquad p = \mathrm{const} > 0,$$

where R is a rational function, then the solution $\tilde{f}$ will have the form of a finite combination of elementary functions and probability integrals.

§ 4. Consider the integro-differential equation

$$\frac{1}{\sqrt{2\pi}}\left(\frac{d}{dx^2} - q^2\right)\int_0^\infty k(x-t)f(t)\,dt = g(x), \qquad x > 0, \quad q > 0, \qquad (7)$$

where k has the properties of the kernel of equation (5), but now $f(x) \in L_2(0,\infty)$, and g is a given generalized function from the class indicated in § 3. The exact meaning of equality (7) is simpler to formulate for the Fourier-transformed functions:

$$-(x^2+q^2)K(x)F^+(x) = G(x) + F^-(x), \qquad -\infty < x < \infty.$$

Here G is the Fourier integral of the function g, equal to zero for $x < 0$; $F^-(x)$ is an unknown function, analytic in the lower half-plane, and

$$\int_{-\infty}^{\infty}\left|\frac{F^-(x+iy)}{x+iy-i}\right|^2 dx < \mathrm{const}, \qquad y \leqslant 0;$$

$F^+(x)$ is the Fourier integral of the function $f(x)$, extended by zero for $x < 0$. Applying Theorem 1, we obtain that, if equation (7) is solvable (we do not write out the solvability condition here), then, in order to construct an approximate solution, it is sufficient to find, in the class $L_2(0,\infty)$, a solution of an equation of the form (7), where in place of f, k, and g there stand the functions $\tilde{f}$, $\tilde{k}$, and $\tilde{g}$. Moreover:

a) $\tilde{k}(x) \in L_2(-\infty,\infty)$, the function $\tilde{K}(x)\sqrt{x^2+q^2}$ satisfies a Hölder condition with exponent not less than 1/2, $\tilde{K}(x)\sqrt{x^2+q^2} \neq 0$; $[\arg\tilde{K}(x) \times \sqrt{x^2+q^2}]_{-\infty}^{\infty} = 0$;

b) $\max|[K(x) - \tilde{K}(x)](x^2+y^2)| < \infty$;

c) the quantity δ, defined in § 3, is less than 1;

d) $G(x) - \tilde{G}(x) \in L_2(-\infty,\infty)$. If conditions c)–d) are fulfilled, then

$$\left(\int_0^\infty |f(x)-\tilde{f}(x)|^2\,dx\right)^{1/2} \leqslant$$

$$\leqslant \frac{\max|\tilde{X}^+(x^2+q^2)^{-1/4}|}{(1-\delta)\min|\tilde{X}^-(x^2+q^2)^{1/4}|}\left(\int_0^\infty \left|\frac{1}{\sqrt{2\pi}}\left(\frac{d^2}{dx^2}-q^2\right)\int_0^\infty k(x-t)\tilde{f}(t)\,dt-g\right|^2 dx\right)^{1/2}$$

§ 5. Analogous results have been obtained for equation (4) in the case when the function $1+K(x)$ has zeros of integer orders at a finite number of points, and also in the case when this function, in a neighborhood of the point $x=0$, has the form $|x|\omega(x)$, where $\omega(x)$ is continuous and nonzero. Some cases of a system of Wiener–Hopf equations have also been investigated.

§ 6. All the cases indicated in §§ 2–5 occur in the solution of problems of mathematical physics belonging to the class described in (4). We give a simple example. Let the equation $u_{xx}=u_t$ be given, where $-\infty<x<\infty$, $0<t<\infty$. Conditions: $u(x,0)=0$ for $x<0$, and $u(x,0)-\lambda u(x,1)=g(x)$ for $x>0$. Using the well-known formula

$$u(x,t)=\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} u(\xi,0)\exp\frac{(x-\xi)^2}{4t}\,d\xi,$$

we arrive at the equation

$$u(x,0)-\frac{\lambda}{2\sqrt{\pi}}\int_0^\infty u(\xi,0)\exp\frac{(x-\xi)^2}{4}\,d\xi=g(x), \qquad x>0.$$

For $\lambda\in[1,\infty)$ we have the case of § 2; if $\lambda\in[1,\infty)$, the first case of § 5. Along with the given problem, approximate solutions have been obtained for a number of other problems.

The range of applications of Theorems 1 and 2 is not limited to Wiener–Hopf equations. These and similar theorems can be applied in those cases when the operator K or the solution f has "singularities" of one sort or another that are annihilated if the differences $K-\tilde{K}$ or $f-\tilde{f}$ are taken. For example, if K is an integro-differential operator, then the differential part may turn out to be its "singularity." The "singularities" of the function f may turn out to be points at which smoothness is violated. This will be the case, for example, when $f\in L_2$, and an approximate solution $\tilde{f}$ is sought such that the function $\tilde{f}-f$ turns out to be differentiable.

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Note: Figure translations are in progress. See original paper for figures.