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Abstract

Full Text

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ON INTEGRAL REPRESENTATIONS OF FINITE GROUPS

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We shall use the following notation: Z is the ring of rational integers; R_p is the ring of p-integral rational integers; J_p is the ring of integral p-adic integers; Q is the field of rational numbers; Ω_p is the field of p-adic numbers; $R(G, T)$ is the group ring of the group G over the ring T.

In ([1]) the following theorem was established:

Theorem 1. *A finite group G has a finite number of indecomposable Z-representations if and only if its order is cube-free and all Sylow p-subgroups of this group are cyclic.*

If the group G satisfies the conditions of Theorem 1, then, generally speaking, there exist indecomposable Z-representations of this group that are not realized in ideals of the integral group ring $R(G, Z)$. For example, as shown in ([2]), a cyclic group of order p^2 ($p \neq 2$) has an indecomposable Z-representation of degree $p^2 + 1$.

We shall say that a finite group G has property (J) if all indecomposable Z-representations of this group are realized in ideals of the group ring $R(G, Z)$.

In the present note the class of finite groups satisfying condition (J) is completely determined. In the second part of the paper a theorem on induced Z-representations is given, generalizing the known theorem on the indecomposability of the integral group ring $R(G, Z)$ of an arbitrary finite group G, and a theorem on the number of irreducible J_p-representations of an arbitrary p-group.

I. **Theorem 2.** *A finite cyclic group G has property (J) if and only if its order is either square-free or equal to 4.*

Theorem 3. *The following conditions are equivalent:*

1) *The group G has property (J).*
2) *The order of the group G is either square-free, or is equal to $4p_1 \cdots p_s$ ($p_1, \dots, p_s$ are odd pairwise distinct primes), and in the latter case the group G simultaneously satisfies the following conditions: a) G is an extension of a normal divisor H of order $p_1 \cdots p_s$ by a cyclic group of order 4; b) G contains no cyclic subgroups of order $4p_i$ ($i = 1, \dots, s$).*

From Theorem 3, in particular, it follows that for a group G of odd order property (J) holds if and only if the order of the group is square-free.

Theorem 3 also admits the following formulation:

A group G has property (J) if and only if all Sylow subgroups of the group G are cyclic and the order of each cyclic subgroup of the group G is either square-free or equal to 4.

The proof of the necessity of the conditions of Theorem 3 is based on Theorem 2. If the group G contains a cyclic subgroup H whose order is divisible by p^2 (p an odd prime) or by $4p$, then the subgroup H has an indecomposable Z-representation Γ that is not realized in ideals of the ring $R(H, Z)$, and with the aid of Γ one can construct an indecomposable Z-representation of the group G which likewise cannot be obtained with the aid of ideals of the ring $R(G, Z)$.

The proof of sufficiency is technically more complicated and is based on the following facts, which are of independent interest.

Theorem 4. Let G be a group with cyclic Sylow p-subgroups (for all $p \mid (G:1)$). If Γ is a Z-representation of the group G, all irreducible Q-components of which are mutually equivalent, then Γ (over Z) decomposes into a sum of irreducible Z-components:

$$\Gamma = \Gamma_1 + \cdots + \Gamma_m.$$

Lemma 1. Let G be a group of square-free order:

$$G: \quad a_1^{p_1} = 1, \dots, a_s^{p_s} = 1; \quad a_i a_j = a_j a_i \quad (i, j = 1, \dots, s);$$

$$b_1^{q_1} = 1, \dots, b_r^{q_r} = 1; \quad b_i b_j = b_j b_i \quad (i, j = 1, \dots, r);$$

$$b_i^{-1} a_j b_i = a_j^{\mu} \quad (p_1, \dots, p_s;\ q_1, \dots, q_r \text{ are distinct primes}).$$

Let

$$\Gamma: \ a \to \tilde{\varepsilon};\ b \to B \quad (a = a_1 \cdots a_s; \quad b = b_1 \cdots b_r)$$

be an irreducible Q-representation of the group G, where $\tilde{\varepsilon}$ is the matrix of the operator of multiplication by a primitive root ε of degree $p_1 \cdots p_r$ in the field $Q(\varepsilon)$. Suppose that the elements $b_1, \dots, b_r$ are numbered so that the elements $b_1, \dots, b_k$ exhaust all elements b_i $(1 \le i \le r)$ that simultaneously satisfy the conditions

$$[b_i, a_1] \ne 1; \qquad [b_i, a_j] = 1 \quad (j = 2, \dots, r).$$

Then there exist exactly $q_1 \cdots q_k$ pairwise inequivalent R_p-representations of the group G that are equivalent over Q to the representation Γ.

Lemma 2. Let the order of the group G be cube-free and let all Sylow p-subgroups of this group be cyclic. Let the $G - J_p$-module M be represented as a direct sum of irreducible $G - J_p$-modules:

$$M = M_1 + \cdots + M_s.$$

If the modules

$$\Omega_p M_1, \dots, \Omega_p M_s$$

are $G - \Omega_p$-isomorphic, then every $G - J_p$-automorphism θ of the module M can be represented in the form of a product

$$\theta = \theta' \theta_1 \cdots \theta_k,$$

where θ' is a diagonal automorphism, and θ_i $(i = 1, \dots, k)$ are elementary triangular automorphisms of this module.

We note that Theorem 4 and Lemma 2 play an essential role in the author’ s proposed proof of Theorem 1.

II. It is known that the integral group ring $R(G, Z)$ is indecomposable into a direct sum of left (right) ideals. This fact may be formulated as follows: the Z-representation of the group G induced by the unit character of the unit subgroup of the group G is indecomposable.

Theorem 5. Let G be an arbitrary finite group, H a subgroup of the group G, χ a linear character of the subgroup H, and let Γ be the representation of the group G over the ring $Z[\chi]^*$, induced by this linear character. Then the representation Γ is indecomposable over the ring $\bar{Z}$ of all algebraic integers.

In particular, the transitive permutation representation of the group G, regarded as a Z-representation of the group G, is indecomposable.

The following theorem, by its methods of proof, is closely connected with Lemma 1.

Theorem 6. Let G be a p-group $(p \neq 2)$; let $\bar{\Gamma}$ be an absolutely irreducible representation of the group G; let χ be the character of the representation $\bar{\Gamma}$; and let Γ be an irreducible representation of the group G over the field Ω_p, corresponding to the representation $\bar{\Gamma}$. Let

$$T = \Omega_p(\varepsilon)$$

be the smallest splitting field of the circle in which the representation $\bar{\Gamma}$ can be written monomially $(T \supset \Omega_p(\chi))$.

Then there exist at least

$$m = (T : \Omega_p(\chi))$$

pairwise inequivalent J_p-representations (and R_p-representations) of the group G that are equivalent over Ω_p to the representation Γ.

Let us point out that it is easy to indicate examples where $m > 1$. For example, for the group

$$G: \quad a^{p^2} = 1; \quad b^p = 1; \quad b^{-1}ab = a^{1+p} \quad (p \neq 2)$$

there exists an absolutely irreducible representation $\bar{\Gamma}$ of degree p, for which $m = p$.

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References

1. S. D. Berman, P. M. Gudivok, *Reports and Communications of Uzhgorod University*, series phys.-math., No. 5, 74 (1962).

2. S. D. Berman, P. M. Gudivok, *Dokl. Akad. Nauk SSSR*, **145**, No. 6, 1199 (1962).

* The ring $Z[\chi]$ is obtained by adjoining to the ring Z all values of the character χ.

Note: Figure translations are in progress. See original paper for figures.

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