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Abstract

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ON TENSOR FIELDS ON THE n-DIMENSIONAL SPHERE

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The main purpose of the present paper is to describe spaces of tensor fields on the n-dimensional sphere S_n in Euclidean space E_{n+1} that are invariant and irreducible with respect to the full orthogonal group $O(n+1)$, and such that any continuous tensor field on S_n admits a uniform approximation by finite sums of fields from these spaces.

This problem for scalar fields was solved long ago, and for vector fields by another method by A. A. Kirillov in (^{2}).

1°. Fix in E_{n+1} a rectangular Cartesian coordinate system with origin O at the center of the unit sphere S_n; its orthonormal frame will also be called **fixed**.

At an arbitrary point a lying on S_n, choose an orthonormal frame so that its $(n+1)$-st vector is directed along the exterior normal to S_n. In what follows the index $n+1$ will be denoted by 0.

Take in E_{n+1} an arbitrary tensor with coordinates

$$T_{i_1,i_2,\dots,i_k} \quad (i_j = 0, 1, \dots, n) \tag{1}$$

with respect to this frame, and choose an integral nonnegative number $l \leq k$. Then the coordinates

$$T_{\alpha_1,\alpha_2,\dots,\alpha_l,\underbrace{0,\dots,0}_{k-l}} \quad (\alpha_j = 1, 2, \dots, n) \tag{2}$$

may be taken as the coordinates of a tensor of valence l on the sphere S_n at this point a. We obtain on the sphere a tensor field whose coordinates we denote by

$$_k\overline{T}_{\alpha_1,\alpha_2,\dots,\alpha_l} = T_{\alpha_1,\alpha_2,\dots,\alpha_l,\underbrace{0,\dots,0}_{k-l}}. \tag{3}$$

The tensor fields $_k\overline{T}_{\alpha_1,\dots,\alpha_l}$ on S_n will be called **induced**, and the original tensor $T_{i_1,i_2,\dots,i_k}$ the preimage of the given induced field. Suppose that the tensor

$T_{i_1,i_2,\dots,i_k}$ induces the tensor field ${}_k\overline{T}_{\alpha_1,\alpha_2,\dots,\alpha_l}$; then the indices $i_{l+1}, i_{l+2}, \dots, i_{l+k}$ will be called the **annihilated** indices of the tensor $T_{i_1,i_2,\dots,i_k}$. (It is possible to annihilate not only the last indices.)

Obviously, in this way one may obtain, in particular, scalar fields on S_n (for $l = 0$).

To every rotation in E_{n+1} of the preimage T there corresponds the corresponding motion on S_n of the induced tensor field ${}_k\overline{T}$.

2°. Suppose that a scalar field on S_n is given in the fixed coordinate system by the function

$$f(x) = C x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}, \tag{4}$$

where a_i are nonnegative integers; this field will be induced, and its preimage is the tensor $T_{i_1,i_2,\dots,i_r}$ $(r = a_0 + a_1 + \dots + a_n)$, whose only coordinate different from zero with respect to the fixed frame is the coordinate

$$T_{\underbrace{0,0,\dots,0}_{a_0},\underbrace{1,1,\dots,1}_{a_1},\dots,\underbrace{n,\dots,n}_{a_n}}, \tag{5}$$

equal to C. Since every continuous function on S_n admits a uniform approximation by polynomials, every continuous scalar field on S_n admits a uniform approximation by finite sums of induced scalar fields.

3°. Define the function $f(x)$ $(-1 \le x \le 1)$

$$\begin{aligned} f(x) &= 1 && \text{for } |x| \ge \frac{1}{2}\frac{1}{\sqrt{n+1}}, \\ f(x) &= \cos^2(2\pi\sqrt{n+1}\,x) && \text{for } \frac{1}{2}\frac{1}{\sqrt{n+1}} \ge |x| \ge \frac{1}{4}\frac{1}{\sqrt{n+1}}, \\ f(x) &= 0 && \text{for } \frac{1}{4}\frac{1}{\sqrt{n+1}} \ge |x|. \end{aligned} \tag{6}$$

On the unit sphere S_n, define $n+1$ functions $f_i(x_0, x_1, \dots, x_n)$ by the formula

$$f_i(x_0, x_1, \dots, x_n) = f(x_i) \Big/ \sum_{j=0}^{n} f(x_j) \qquad (i = 0, 1, \dots, n). \tag{7}$$

Take in E_{n+1} all tensors of the given valency k whose coordinates vanish if at least one of their indices assumes a fixed value i in the fixed frame. From them one can choose n^k linearly independent ones. The tensor fields of valency k induced by these tensors $(l = k)$ form, obviously, at each point on S_n, except the points $x_i = 0$, a complete basis of linearly independent tensors of valency k. Denote them by

$$ {}_{ilk}P_{\alpha_1,\alpha_2,\dots,\alpha_k} \qquad (l = 1, 2, \dots, n^k). \tag{8} $$

4°. **Theorem 1.** *Every continuous tensor field on the sphere S_n admits a uniform approximation by finite sums of induced tensor fields.*

Proof. Every continuous tensor field $Q_{\alpha_1,\alpha_2,\dots,\alpha_k}$ of valency k on S_n can be represented in the form of the sum

$$ Q_{\alpha_1,\alpha_2,\dots,\alpha_k} = \sum_{i=0}^{n} f_i(x_0, x_1, \dots, x_n)\, Q_{\alpha_1,\alpha_2,\dots,\alpha_k}. \tag{9} $$

Since the tensor fields (8) on S_n, for fixed i, are continuous in the domain $|x_i| > 0$ and form a complete basis, the field $f_i Q_{\alpha_1,\alpha_2,\dots,\alpha_k}$, for $|x_i| \geq$

$$ \geq \frac{1}{4\sqrt{n+1}}, $$

is uniquely decomposed with respect to these basis tensor fields:

$$ f_i Q_{\alpha_1,\alpha_2,\dots,\alpha_k} = \sum_{l=1}^{n^k} {}_{ilk}P_{\alpha_1,\alpha_2,\dots,\alpha_k}\, C_{il}, \tag{10} $$

where C_{il} are continuous scalar functions defined on S_n for

$$ |x_i| \geq \frac{1}{4\sqrt{n+1}} $$

and vanishing when

$$ |x_i| = \frac{1}{4\sqrt{n+1}}; $$

for

$$ |x| < \frac{1}{4\sqrt{n+1}} $$

we put $C_{il} = 0$.

We already know that the scalar fields $C_{il}(x)$ admit a uniform approximation in the form of sums of scalar fields induced from scalar fields. It is obvious that the tensor product of the preimage of the tensor field ${}_{il}P_{\alpha_1,\alpha_2,\dots,\alpha_k}$ by the preimages of these scalar fields will give tensors which will induce fields of valence k uniformly approximating the tensor field ${}_{il}P_{\alpha_1,\alpha_2,\dots,\alpha_k} C_{il}$. Thus the tensor fields $f_i Q_{\alpha_1,\dots,\alpha_k}$, and consequently also $Q_{\alpha_1,\dots,\alpha_k}$, admit uniform approximation by induced tensor fields. This proves Theorem 1.

5°. Take an irreducible, with respect to rotations in E_{n+1} (the full orthogonal group $O(n+1)$), linear space Π of tensors T in E_{n+1}; it induces on the sphere S_n a space $\overline{\Pi}$ of tensor fields $\overline{T}$, and it is assumed that the same indices are annulled for all tensors T. The space Π is invariant with respect to the full group of motions on S_n; moreover Lemma 1 holds.

Lemma 1. *The preimage of a tensor either fills* Π, *or is itself equal to* 0*; that is, either the whole space is mapped to* 0, *or the mapping* $\Pi \to \overline{\Pi}$ *is one-to-one and the dimensions of* Π *and* $\overline{\Pi}$ *coincide. In the latter case the representation of* $O(n+1)$ *in the space* $\overline{\Pi}$ *is equivalent to the representation in the space* Π.

6°. It is known (^{1}) that any tensor T in E_{n+1} can be represented as a finite sum of tensor products of the form

$$g_{i_1,i_2} \cdot g_{j_1,j_2} \cdots g_{m_1,m_2} \cdot r_{l_1,l_2,\dots,l_k}, \tag{11}$$

where g_{p_1,p_2} is the metric tensor, and $r_{l_1,l_2,\dots,l_k}$ is a tensor obtained according to a Young diagram, whose contraction in any two indices with the metric tensor gives zero (a "traceless" tensor) and the sum of the lengths of the first two columns is less than or equal to $n+1$. We shall call such Young tensors **admissible**.

As the Young symmetrizer, in contrast to (^{1}), we take the symmetrizer in which alternation over columns is first carried out, and then symmetrization over rows of the Young diagram.

The metric tensor g_{ij}, obviously, can induce only: a) the field of the metric tensor on S_n ($l=2$); b) the zero vector field ($l=1$); c) the unit scalar field ($l=0$).

Young admissible tensors with one and the same diagram form, with respect to the full group $O(n+1)$, an irreducible space Π of tensors T.

Lemma 2. *The tensor field* $\overline{T}$ *on* S_n*, induced in any way by a given admissible tensor* T*, is representable in the form of a linear combination of tensor fields* $\overline{T}_{(1)}, \overline{T}_{(2)}$*, which, up to a substitution of indices, are induced by the same tensor* T *under the condition that the annulled indices are taken only from the first row.*

Lemma 2 is easily proved by induction on the number of annulled indices outside the first row, taking into account that if the first row of the Young diagram is symmetrized with one more index, the result is zero.

We can now formulate Theorem 1 more precisely.

Theorem 1 (second formulation). *Every continuous tensor field on* S_n *admits uniform approximation by sums of tensor products of the metric tensor* *on S_n, taken any number of times, with a tensor field induced by an admissible tensor T with annulled indices only in the first row.*

7°. Take the tensor

$$T_{i_{11},i_{12},\ldots,i_{1s_1},i_{21},\ldots,i_{ms_m}}, \tag{12}$$

where the indices with identical first indices form symmetric groups of indices of the Young diagram.

Let us annihilate the r indices of the first row. The resulting induced tensor field on S_n may be denoted by

$$T_{\underbrace{0,0,\ldots,0}_{r}\,\alpha_{1,r+1},\alpha_{1,r+2},\ldots,\alpha_{1s_1},\alpha_{21},\ldots,\alpha_{ms_m}}. \tag{13}$$

One can compute that the covariant derivative of this tensor has the form

$$\begin{aligned} T_{\ldots,\beta} = {} & rT_{\underbrace{0,0,\ldots,0}_{r-1},\beta,\alpha_{1,r+1},\ldots,\alpha_{1s_1},\ldots,\alpha_{ps_p}} \\ & - \sum_{q=r+1}^{s_1} g_{\beta,\alpha_{1q}}\, T_{\underbrace{0,\ldots,0}_{r}\,\alpha_{1,r+1},\ldots,\alpha_{1,q-1},0,\alpha_{1,q+1},\ldots,\alpha_{1s_1},\ldots,\alpha_{ms_m}} \\ & - \sum_{t=2}^{m}\sum_{q=1}^{s_t} g_{\beta,\alpha_{tq}}\, T_{\underbrace{0,\ldots,0}_{r}\,\alpha_{1,r+1},\ldots,\alpha_{1s_1},\ldots,\alpha_{tq-1},0,\alpha_{tq+1},\ldots,\alpha_{ms_m}}. \end{aligned} \tag{14}$$

If one takes the tensor described in (1) on p. 214, then it is easy to see that, after the action of the Young symmetrizer on it, it will have a coordinate with indices without $(*)$ that is different from zero. Consequently, when all indices of the first row are annihilated at the point $x_1 = 1$ (on S_n in a fixed coordinate system), it will give a nonzero coordinate of the induced field. Hence we obtain:

Lemma 3. *The space Π of tensor fields $\overline{T}$ on S_n, induced by the admissible irreducible space Π of tensors T with a given Young diagram by annihilating all indices of the first row, is nonempty.*

On the sphere S_n these tensor fields $\overline{T}$ form a space $\overline{\Pi}$ irreducible with respect to the group $O(n+1)$ of motions. They are traceless and are Young tensors with a diagram differing from the Young diagram of the preimages by the absence of the first row. Such spaces $\overline{\Pi}$ we shall call **fundamental**.

From formula (14) the refinement of Theorem 1 easily follows:

Theorem 2. *The set of fundamental tensor fields is complete in the sense that any tensor field on S_n admits a uniform approximation by finite sums of fundamental fields and their covariant derivatives, multiplied, possibly, a certain number of times by the metric tensor field.*

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REFERENCES

1. H. Weyl, *The Classical Groups*, Moscow, 1947.
2. A. A. Kirillov, *DAN*, **116**, No. 6, 538 (1957).

Note: Figure translations are in progress. See original paper for figures.

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