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Abstract

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INTERPOLATION AND EXTRAPOLATION OF LINEAR OPERATORS IN ORLICZ SPACES

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A function M is called an N-function if

$$M(u) = \int_0^{|u|} p(t)\,dt,$$

where $p(t)$ is a nondecreasing positive function; $N(u) = \int_0^{|u|} q(t)\,dt$, where $q(t)$ is the right inverse of p, is called complementary to $M(u)$ (^{9}).

Let D be a measurable set (possibly of infinite measure) in n-dimensional Euclidean space E_n. Then by the class $L_M(D)$ we mean the set of functions u for which

$$\int_D M[u(P)]\,dP < \infty.$$

To the Orlicz space $L_M^*(D)$ we assign those functions u satisfying the condition

$$\int_D |u|\,|v|\,dP < \infty$$

for all $v \in L_N(D)$. The norm in the space $L_M^*(D)$ is introduced by the equality

$$\|u\|_M = \sup_v \int_D u(P)v(P)\,dP, \qquad \int_D N[v(P)]\,dP \leq 1. \tag{1}$$

Besides N-functions, we shall need one more class of functions. An absolutely continuous positive function $M(u)$ will be called quasipower if

$$\alpha = \sup_u \frac{uM'(u)}{M(u)} < \infty, \qquad \beta = \inf_u \frac{uM'(u)}{M(u)} > -\infty;$$

these numbers α and β, determined by the function M, will be denoted by $\alpha(M)$ and $\beta(M)$. We also define the numbers $\alpha_\infty(M)$ and $\beta_\infty(M)$ by the equalities

$$\alpha_\infty(M) = \overline{\lim}_{u\to\infty}\frac{uM'(u)}{M(u)}; \qquad \beta_\infty(M) = \underline{\lim}_{u\to\infty}\frac{uM'(u)}{M(u)}.$$

If $\beta(M) > 1$, then M, although generally speaking not an N-function, has associated with it an N-function

$$\widetilde{M}(u) = \int_0^{|u|}\frac{M(t)}{t}\,dt,$$

satisfying the inequalities

$$\frac{1}{\alpha(M)}M(u) \le \widetilde{M}(u) \le \frac{1}{\beta(M)}M(u).$$

Moreover, $\widetilde{M}$ is a quasipower function and

$$\alpha(\widetilde{M}) \le \alpha(M), \qquad \beta(\widetilde{M}) \ge \beta(M);$$

analogous inequalities are also valid for $\alpha_\infty, \beta_\infty$. Let $\bar{D}$, generally speaking different from D, be a measurable set in $\bar{n}$-dimensional Euclidean space. For brevity we introduce the notation

$$\rho(D, M, f) = \int_D M[|f(P)|]\,dP.$$

We have obtained the following interpolation theorem 1.

Theorem 1. *Let the linear operator K act from $L_{p_i}(D)$ into $L_{\bar{p}_i}(\bar{D})$ $(1 \le p_1 \le \bar{p}_1,\ 1 \le p_2 \le \bar{p}_2)$, and let M be a quasipower function, with $p_1 < \beta(M) \le \alpha(M) < p_2$, while $\bar{M}$ is related to M by the relation $\bar{M}^{-1}(t) =$*

$$= t^{\delta}M^{-1}\left(t^{\Delta/\bar{\Delta}}\right)\left(\Delta = \pi_1 - \pi_2, \quad \bar{\Delta} = \bar{\pi}_1 - \bar{\pi}_2, \quad \delta = \frac{\pi_1\bar{\pi}_2 - \pi_2\bar{\pi}_1}{\pi_1 - \pi_2}, \quad \pi_i = \frac{1}{p_i}, \quad \bar{\pi}_i = \frac{1}{\bar{p}_i}\right).$$

Then

$$\rho(\bar{D}, \bar{M}, Kf) \le T[\rho(D, M, f)],$$

where T is a monotonically increasing function depending on $p_1, \bar{p}_1, p_2, \bar{p}_2, M, \|K\|_{(L_{p_i}(D)\to L_{\bar{p}_i}(\bar{D}))}$ $(i = 1, 2)$, and the dependence on the norms $\|K\|_{p_i\to\bar{p}_i}$ is such that, when they tend to zero and the other constants are fixed, $T(u) \to 0$ $(u(>0)$ is any number).

From Theorem 1 follows the interpolation theorem 2 for Orlicz spaces.

Theorem 2. Suppose the conditions of Theorem 1 are satisfied and, in addition, the function M is an N-function. Then the operator K also acts from $L^*_M(D)$ into $L^*_{\widetilde{M}}(\bar{D})$, where

$$\widetilde{M}(u) = \int_0^{|u|} \frac{\bar{M}(t)}{t}\, dt,$$

and

$$\|K\|_{(L^*_M(D) \to L^*_{\widetilde{M}}(\bar{D}))} \to 0$$

when

$$\|K\|_{(p_i \to \bar{p}_i)} \to 0,$$

while the remaining quantities remain constant.

In the case when at one of the endpoints p_1 or p_2 the operator K is also completely continuous, there holds a result analogous to that of M. A. Krasnosel' skii (^{3}) for the spaces L_p.

Theorem 3. Let the linear operator K act from $L_{p_1}(D)$ into $L_{\bar{p}_1}(\bar{D})$ or from $L_{p_2}(D)$ into $L_{\bar{p}_2}(\bar{D})$ as a completely continuous operator, and at the other endpoint be simply continuous. Then K also acts as a completely continuous operator from $L^*_M(D)$ into $L^*_{\widetilde{M}}(\bar{D})$.

In the case when $\operatorname{mes} D < \infty$, Theorems 1, 2, and 3 remain valid under weaker assumptions on M: instead of the inequalities for $\alpha(M)$, $\beta(M)$, it is required that the corresponding inequalities hold for $\alpha_\infty(M)$, $\beta_\infty(M)$.

The first interpolation theorems of the theory of linear operators in the spaces L_p are due to M. Riesz (^{1}). A further development of this theory was given by C. G. Krein (^{6}). Then A. P. Calderón and A. Zygmund (^{2}) and S. Koizumi (^{7}) proved an interpolation theorem in Orlicz spaces for the case $p_i = \bar{p}_i$. M. A. Krasnosel' skii (^{3}) obtained, for the spaces L_p, a theorem according to which the complete continuity of an operator K at one of the endpoints L_{p_1} or L_{p_2} implies the complete continuity of the operator K in the intermediate spaces L_p ($p_1 < p < p_2$). Theorems 2 and 3 proved by us generalize these results.

At the basis of the proof of Theorem 1 and of Theorem 4 formulated below lies the idea of A. P. Calderón and A. Zygmund (^{10}) of using an estimate of the measure $E_y(Kf)$ ($P \in E_y(Kf)$, if $|Kf| > y$). This idea was realized by them in the case of singular integrals of a special form in order to prove their boundedness in the spaces L_p. A careful analysis of the work (^{10}) has allowed us to prove the following result:

Theorem 4 (extrapolation). Let the integral operator

$$Af = \int_D A(P, Q) f(Q)\, dQ$$

possess the following properties:

1) A is bounded as an operator acting from $L_{p_0}(D)$ into $L_{p_0}(\bar{D})$.
2) The kernel $A(P,Q)$ satisfies the condition: for any concentric spheres S and S_1 with ratio of radii $2/1$, for any Q and Q' from S_1 the inequality

$$\int_{S'\cap D} |A(P,Q) - A(P,Q')|\, dP \leq B < \infty, \tag{2}$$

holds,

where S' is the complement of S, and the constant B does not depend on S, Q, or Q'.

I. Then the operator A is bounded as an operator acting from $L_M^*(D)$ into $L_M^*(D)$. Here the N-function M satisfies the condition

$$1 < \beta \leq \frac{uM'(u)}{M(u)} \leq \alpha \tag{3}$$

for $\alpha < p_0$. In this case the norm of the operator $\|A\|_M$ depends only on $\|A\|_{p_0}, \alpha, \beta, p_0, D, B$.

II. If instead of condition 2) the condition

$$2')\quad \int_{S'\cap D} |A(P,Q) - A(P',Q)|\, dQ \leq B' < \infty,$$

where $P, P' \in S_1$, is satisfied, and B' does not depend on P, P', S, then the operator A is bounded in the spaces $L_M^*(D)$ for $\beta > p_0$. In this case the norm of the operator depends only on $\|A\|_{p_0}, \alpha, \beta, p_0, D, B'$.

III. If conditions 1), 2), 2') are satisfied simultaneously, then the operator A is bounded in all spaces L_M^* satisfying condition (3), and $\|A\|_M$ depends only on $\alpha, \beta, p_0, \|A\|_{p_0}, D, B, B'$.

Let us note that the main assertion of the theorem is not the boundedness of the operator A, but the fact that its norm depends only on the constants occurring in the hypotheses of the theorem. This is connected with the fact that an extrapolation theorem is usually applied according to the following scheme.

There is a sequence of integral operators

$$A_m f = \int_D A_m(P,Q) f(Q)\, dQ,$$

converging strongly in some space L_{p_0} to the operator A (most often $p_0 = 2$). In order to establish the boundedness of the operator A in other spaces L_p and in Orlicz spaces, it is necessary to verify whether the estimates of the extrapolation theorem hold uniformly with respect to m. In this way one can prove boundedness in Orlicz spaces of a certain kind for singular integrals, proved by the author earlier in paper (^{8}). The application of the extrapolation theorem considerably simplifies the proof.

We give two more results characterizing the operator A in the limiting cases $p=1$ and $p=\infty$.

Theorem 5. If $\operatorname{mes} D<\infty$ and the operator A satisfies conditions 1), 2'), then for every measurable bounded function f there exists a number $\delta>0$ such that

$$I=\int_D \exp(\delta|Af|)\,dP<\infty, \tag{4}$$

where δ depends only on $\operatorname{vrai\,sup}|f|, B', \|A\|_{p_0}, p_0, n$; I also depends on $\operatorname{mes} D$.

This theorem follows from the estimate $\|A\|_p \leq cp$ for $p\geq 2p_0$, and from a result of V. I. Yudovich (^{15}) concerning operators with such growth in p.

By passing to the conjugate operator, Theorem 3 implies

Theorem 6. If the operator A satisfies conditions 1) and 2) of the extrapolation theorem and $\operatorname{mes} D<\infty$, then the estimate

$$\int_D |A(f)|\,dP \leq \int_D \ln\left(\frac{|f|}{c_1}+1\right)\left(\frac{|f|}{c_1}+1\right)\,dP+c_2, \tag{5}$$

holds, where c_1, c_2 are constants depending only on $\|A\|_{p_0}, p_0, \operatorname{mes} D, B, n$.

The theorems proved make it possible to obtain a number of new results. We present some of them.

Corollary 1. Potential-type integrals over the whole n-dimensional space E_n

$$Kf\int_D \frac{f(Q)}{r^\lambda(P,Q)}\,dQ$$

were studied by S. L. Sobolev (^{4}).

They established that the operator K acts from $L_p\left(1<p<\dfrac{n}{n-\lambda}\right)$ into $L_{\bar p}$, where

$$\frac{1}{p}-\frac{1}{\bar p}=\frac{n-\lambda}{n}.$$

Let $M(u)$ satisfy the condition

$$1 < \beta \leq \frac{uM'(u)}{M(u)} \leq \alpha < \frac{n}{n-\lambda};$$

then, on the basis of the interpolation theorem, the operator K acts as a bounded operator from L_M^* into $L_{\widetilde{M}}^*$ (the connection of M with $\widetilde{M}$ is indicated in Theorem 3). If integration is carried out not over the whole space, but over a measurable subset of finite measure, then the operator K, in addition, is completely continuous. It is understood that Theorems 2 and 3 also make it possible to obtain embedding theorems in Orlicz spaces.

Some results on the boundedness of integrals of potential type for domains of finite measure and embedding theorems were obtained by I. V. Gelman ([16,17]).

Corollary 2. Let the operator K be bounded as an operator acting from $L_{p_1}(D)$ into $L_{p_1}(D)$ and from $L_{p_2}(D)$ into $L_{p_2}(D)$, and let N be its right (left) regularizer* in one of the spaces $L_{p_1}(D)$ or $L_{p_2}(D)$, with N also bounded in both spaces L_{p_1} and L_{p_2}. Then, on the basis of Theorem 3, we obtain that the operator N regularizes K on the right (left) also as an operator acting from L_M^* into L_M^*, where $M(u)$ satisfies the condition

$$p_1 < \beta \leq \frac{uM'(u)}{M(u)} \leq \alpha < p_2.$$

Corollary 3. Since regularization is the basis of the investigations of S. G. Mikhlin ([11–14]) on multidimensional singular integral equations, all these investigations, carried out in the spaces $L_p(D)$, automatically extend to Orlicz spaces**.

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* The operator N regularizes K on the right (left) if $KN = I + B$ ($NK = I + B$), where B is completely continuous.

** When $\operatorname{mes} D < \infty$, it is sufficient to require that, instead of the inequalities for α, β, the corresponding inequalities for α_∞ and β_∞ be fulfilled. This remark also applies to Theorem 4 (the extrapolation theorem).

Note: Figure translations are in progress. See original paper for figures.

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