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# B. D. ROMM

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**Abstract**

**Full Text**

**B. D. ROMM**

**DECOMPOSITION INTO IRREDUCIBLE REPRESENTATIONS OF THE TENSOR PRODUCT OF TWO IRREDUCIBLE REPRESENTATIONS OF THE REAL UNIMODULAR GROUP OF SECOND ORDER (THE CASE OF TWO DISCRETE SERIES)**

*(Presented by Academician P. S. Novikov, 12 VI 1963)*

The real unimodular group of second order  $G$  is realized, as in <sup>(1)</sup>, in the form of the group  $G_0$  of all unimodular complex matrices of second order leaving invariant the form  $z_1\bar{z}_1 - z_2\bar{z}_2$ . The groups  $G$  and  $G_0$  are isomorphic. The isomorphism is established by the formula

$$g = tat^{-1}, \tag{1}$$

where  $a \in G_0$ ,  $g \in G$ ,

$$t = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \tag{2}$$

The irreducible representations of the discrete series are constructed in spaces of analytic functions on the unit disk of the complex plane. Let  $p = \frac{1}{2}, 1, \frac{3}{2}, \dots$ ; let  $\mathfrak{H}$  denote the set of all polynomials  $f(z) = a_0 + a_1z + \dots + a_nz^n$  of arbitrary degrees. On the linear set  $\mathfrak{H}$  we define a system of scalar products  $(f, g)_p$  for  $f$  and  $g \in \mathfrak{H}$  by

$$(f, g)_p = \lim_{k \rightarrow p+0} \frac{2k-1}{\pi} \int_{|z|<1} \overline{f(z)} g(z) (1-z\bar{z})^{2k-2} dz; \tag{3}$$

$$dz = dx dy, \quad \text{for } x = \text{Re } z, \quad y = \text{Im } z, \quad p = \frac{1}{2}, 1, \frac{3}{2}, \dots$$

The closure of the set  $\mathfrak{H}$  in the scalar product (3) forms a Hilbert space, which we denote by  $\mathfrak{H}_l$ ,  $l = 2p$ . In the space  $\mathfrak{H}_l$  we define a unitary representation  $D_p^+$  of the group  $G_0$  by the formula <sup>(1)</sup>

$$T^+(a)f(z) = (-\beta z + \bar{\alpha})^{-2p} f\left(\frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}}\right), \quad a = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G_0. \tag{4}$$

If  $a \in G_0$ , then denote by  $\bar{a}$  the matrix

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \beta & \alpha \end{pmatrix}.$$

Put

$$T^-(a) = T^+(\bar{a}). \quad (5)$$

The operators  $T^-(a)$  in the space  $\mathfrak{H}_l$  define a certain unitary representation of the group  $G_0$ , which we denote by  $D_p^-$ .

In the paper we consider tensor products of two representations of the type  $D_p^+$  and  $D_{p_1}^+$ , as well as  $D_p^-$  and  $D_{p_1}^-$ . The first of these we denote by  $D_{pp_1}^+$ , and the second by  $D_{pp_1}^-$ . The representations  $D_{pp_1}^+$  and  $D_{pp_1}^-$  are realized in the tensor product  $\mathfrak{H}_{l_1}$  of the spaces  $\mathfrak{H}_l$  and  $\mathfrak{H}_{l_1}$ ,  $l_1 = 2p_1$ , which is obtained as a closure of the set  $\mathfrak{H}''$  of all polynomials in  $z$  and  $z_1$  in the scalar product

$$\begin{aligned} (f, g)_{pp_1} = & \lim_{\substack{k \rightarrow p+0 \\ k_1 \rightarrow p_1+0}} \frac{(2k-1)(2k_1-1)}{\pi^2} \iint_{\substack{|z|<1 \\ |z_1|<1}} \overline{f(z_1, z_2)} g(z_1, z_2) (1 - z_1 \bar{z}_1)^{2k-2} \times \\ & \times (1 - z_2 \bar{z}_2)^{2k_1-2} dz_1 dz_2. \end{aligned} \quad (6)$$

The series of operators

$$\begin{aligned} K_m(z, z_1, z_2) = & \frac{1}{m!} \{ (l + l_1 + 2m - 1)! [(l - 1)! (l_1 - 1)! \times \\ & \times \sum_{p=0}^m \frac{1}{(l - 1 + p)! (l_1 - 1 + m + p)! p! (m - p)!} ]^{-1} \}^{1/2} \exp \left\{ z \left( \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right) \right\} \end{aligned} \quad (7)$$

converges strongly in the space  $\mathfrak{H}_{l_1}$  on the set  $\mathfrak{H}''$  for any  $m = 0, 1, 2, \dots$  for each fixed  $z$ .

For any fixed  $z_1$  and  $z_2$ , the series of operators

$$K'_m(z, z_1, z_2) = \frac{(l + l_1 + 2m - 1)!}{m!} \{ (l + l_1 + 2m - 1)! [(l - 1)! (l_1 - 1)! \times$$

$$\times \left. \sum_{k=0}^m \frac{1}{(l-1+k)!(l_1-1+m+k)!k!(m-k)!} \right]^{-1} \Bigg\}^{1/2} \times \sum_{n=0}^{\infty} \frac{(z_1^2 \partial / \partial z_1 + z_2^2 \partial / \partial z_2 + lz_1 + l_1 z_2)^n (z_1 - z_2)^m}{(l+l_1+2m+n-1)!n!} \frac{\partial^n}{\partial z^n} \quad (8)$$

converges strongly in the space  $\mathfrak{H}_p$  on  $\mathfrak{H}$ ,  $p = \frac{1}{2}, 1, \frac{3}{2}, \dots$

**Theorem.** Let  $f(z_1, z_2) \in \mathfrak{H}''$ . Then all functions  $\varphi_m(z)$ , where  $m = 0, 1, 2, \dots$ , of the form

$$\varphi_m(z) = \frac{(l-1)(l_1-1)}{\pi^2} \iint_{\substack{|z_1| < 1 \\ |z_2| < 1}} \overline{(z_1 - z_2)}^m (1 - z_1 \bar{z}_1)^{l-2} (1 - z_2 \bar{z}_2)^{l_1-2} \times \\ \times K_m(z, z_1, z_2) f(z_1, z_2) dz_1 dz_2 \quad (9)$$

belong to  $\mathfrak{H}$ . Conversely, let a finite number of the polynomials  $\varphi_m(z)$ ,  $m = 0, 1, 2, \dots$ , be different from zero. Then the formula

$$f(z_1, z_2) = \sum_{m=0}^{\infty} \frac{l+l_1+2m-1}{\pi} \int_{|z| < 1} (1 - z \bar{z})^{l+l_1+2m-2} K'_m(z, z_1, z_2) \varphi_m(z) dz \quad (10)$$

is the inverse of the mapping (9). This mapping admits a closure, where  $f(z_1, z_2) \in \mathfrak{H}_{l_1}$ ,  $\varphi_m(z) \in \mathfrak{H}_{2(p+p_1+m)}$ , and

$$(f, f)_{pp_1} = \sum_{m=0}^{\infty} (\varphi_m, \varphi_m)_{p+p_1+m}. \quad (11)$$

When  $f(z_1, z_2)$  is transformed according to the representation  $D_{pp_1}^+$  ( $D_{pp_1}^-$ ),  $\varphi_m(z)$  is transformed according to  $D_{p+p_1+m}^+$  ( $D_{p+p_1+m}^-$ ).

Formula (11) is the Plancherel formula.

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## REFERENCES

1. V. Bargmann, Ann. Math., **48**, 568 (1947).

*Note: Figure translations are in progress. See original paper for figures.*

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