ON ALGORITHMIC PROBLEMS IN PARTIALLY ORDERED GROUPS

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Abstract

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The article considers questions connected with the solution of algorithmic problems arising in the partial ordering of finitely presented groups (I), and questions of the solvability of algorithmic problems arising when partially ordered groups are specified by a finite number of generators and a finite number of defining inequalities (II) (an exact description of such a specification of partially ordered groups will be given below). In accordance with this, the article is divided into two parts.

I. Let $\mathfrak{A}$ be a group given by a finite number of generators and defining relations. A set of elements of $\mathfrak{A}$ closed with respect to the group operation is a subsemigroup of the group $\mathfrak{A}$. A subsemigroup of the group $\mathfrak{A}$ will be called **invariant** if, for every word P of this subsemigroup, the word $X^{-1}PX$, where X is any word from $\mathfrak{A}$, also belongs to this subsemigroup. An invariant subsemigroup $\widetilde{\mathfrak{A}}^{+}$ of the group $\mathfrak{A}$ will be called **proper** if among its elements there is no 1 of the group $\mathfrak{A}$. The subsemigroup $\mathfrak{A}^{+}=\widetilde{\mathfrak{A}}^{+}\cup 1$ of the group $\mathfrak{A}$ will be called **positive**. If in the group $\mathfrak{A}$ there is at least one positive subsemigroup, then, as is known ([2]), the group $\mathfrak{A}$ can be partially ordered in the following way: $a\leqslant b$ (a,b are elements of the group $\mathfrak{A}$) if and only if the element ba^{-1} or $a^{-1}b$ belongs to the given positive subsemigroup. If for any elements $a,b\in\mathfrak{A}$ there exists an element c such that $a\leqslant c$ and $b\leqslant c$, then the group $\mathfrak{A}$ is called **directed**.

In the author' s paper ([3]) the notion of an **effectively partially orderable** (e.p.o.) group was introduced: a group $\mathfrak{A}$ is called e.p.o. with respect to a given positive subsemigroup $\mathfrak{A}^{+}$ if the problem of membership of words from $\mathfrak{A}$ in $\mathfrak{A}^{+}$ is decidable, i.e. there exists an algorithm determining, for every word from $\mathfrak{A}$, whether it belongs to $\mathfrak{A}^{+}$ or not. A **directed e.p.o. group** will mean an e.p.o. group for which, for any two words A and B, one can effectively construct a word C greater than the words A and B. It is not difficult to see that for e.p.o. groups the word identity problem is decidable.

In the work of P. S. Novikov ([1]) finitely presented groups with the so-called regular system of passing letters are considered; we shall denote them by $\mathfrak{A}_p$. The subsemigroup of the group $\mathfrak{A}_p$ generated by the set of words $X^{-1}Ap_{iBX}$, where p_i are passing letters of the group $\mathfrak{A}_p$, X are arbitrary words from $\mathfrak{A}_p$, and A,B are all possible words from $\mathfrak{A}_p$ not containing passing letters, is a proper subsemigroup. We shall denote this subsemigroup by $\widetilde{\mathfrak{A}}_p^{+}$, and the positive subsemigroup $\widetilde{\mathfrak{A}}_p^+ \cup 1$ by $\mathfrak{A}_p^+$.

Consider the equation in the group $\mathfrak{A}_p$

$$Z_0 Y_1 Z_1 \dots Y_{kZ} k = 1, \tag{1}$$

where Z_i are arbitrary words from $\mathfrak{A}_p$, for which solutions are considered only in words without passing letters.

If there exists an algorithm deciding, for every equation (1), whether it has a solution or not, then we shall say that the **solvability problem for equations (1) is decidable**.

Theorem 1. *If in the group $\mathfrak{A}_p$ the solvability problem for equations (1) is decidable, then in this group the word membership problem for words of the group $\mathfrak{A}_p$ in the pure subsemigroup $\mathfrak{A}_p^+$ of this group is decidable.*

The proof of Theorem 1 and of the other assertions of this paper cannot be given within the limits of the present article. However, we shall describe the algorithm for membership of words of the group $\mathfrak{A}_p$ in the pure subsemigroup $\mathfrak{A}_p^+$.

The number equal to the sum of all exponents for each passing letter p_i is an invariant of the transformation of a word from $\mathfrak{A}_p$ into any word equal to it.

Let, for some word γ, $k_1, k_2, \dots, k_t$ be the sums of the exponents at the passing letters p_i $(i = 1, 2, \dots, t)$ of the group $\mathfrak{A}_p$. If for at least one p_i, $k_i < 0$, or all $k_i = 0$, then it is obvious that $\gamma \notin \mathfrak{A}_p^+$.

Suppose that for the word γ,

$$k_1 + k_2 + \dots + k_t = k > 0$$

and $k_i \geq 0$ $(i = 1, 2, \dots, t)$. Consider all possible representations of the word γ in the form

$$Z_0 p Z_1 p \dots p Z_k, \tag{2}$$

where the passing letters, for simplicity, are denoted without indices. In the representation (2) of the word γ, k passing letters are chosen so that, if the sum of the exponents at the passing letter p_i is equal to k_i, then k_i letters p_i are written out. The segments between adjacent passing letters written in (2) in the word γ are denoted by Z_i.

To each representation (2) of the word γ we assign an equation of the form (1):

$$Z_0 Y_1 Z_1 Y_2 \dots Y_{kZ} k = 1. \tag{3}$$

If for at least one representation (2) of the word γ equation (3) is solvable, then $\gamma \in \mathfrak{A}_p^+$. If, however, for no representation (2) of the word γ does equation (3) have a solution, then $\gamma \notin \mathfrak{A}_p^+$.

Theorem 2. *If in the group $\mathfrak{A}_p$: a) the solvability problem for equations (1) is decidable and b) the identity problem is decidable, then the group $\mathfrak{A}_p$ can be effectively partially ordered by its proper subsemigroup $\mathfrak{A}_p^+ = \mathfrak{A}_p^+ \cup 1$. In this case the group $\mathfrak{A}_p$ is a directed e.i.u. group with respect to the proper subsemigroup $\mathfrak{A}_p^+$.*

Consider the free product

$$\mathfrak{A}'_p = \mathfrak{A}_p * \{\bar{p}\},$$

where $\{\bar{p}\}$ is an infinite cyclic group. To the system of passing letters of the group $\mathfrak{A}'_p$, besides those contained in the group $\mathfrak{A}_p$, we also adjoin the letter $\bar{p}$.

Theorem 3. *In order that the group $\mathfrak{A}'_p = \mathfrak{A}_p * \{\bar{p}\}$ can be effectively partially ordered by the proper subsemigroup $\mathfrak{A}'^+_p = \mathfrak{A}_p^+ \cup 1$, it is necessary and sufficient that in the group $\mathfrak{A}_p$: a) the solvability problem for equations (1) be decidable and b) the identity problem be decidable. Moreover, if for the group $\mathfrak{A}_p$ conditions a) and b) are satisfied, then the group $\mathfrak{A}'_p$ is a directed e.i.u. group with respect to the subsemigroup $\mathfrak{A}'^+_p$.*

From Theorem 3 it follows:

Theorem 4. *There exists a finitely presented directed e.i.u. group with an undecidable conjugacy problem.*

In Theorem 4 one considers the group

$$\mathfrak{A}'_{p_1} = \mathfrak{A}_{p_1} * \{\bar{p}\},$$

where $\mathfrak{A}_{p_1}$ is a group with an undecidable conjugacy problem, constructed by P. S. Novikov [1]. As the proper subsemigroup of the group $\mathfrak{A}'_{p_1} * \{\bar{p}\}$ one takes the subsemigroup

$$\mathfrak{A}'^+_{p_1} \cup 1,$$

the pure subsemigroup of which $\mathfrak{A}'^+_{p_1}$ is generated by the words

$$X^{-1} A \bar{p} B X,$$

where the words A and B do not contain the letter $\bar{p}$.

Theorem 5. *There exists a finitely presented group $\mathfrak{A}_0$ with decidable identity problem and with such a proper subsemigroup $\mathfrak{A}_0^+$,*

that the group $\mathfrak{A}_0$ cannot be effectively partially ordered by this semigroup $\mathfrak{A}_0^+$.

The proof of Theorem 5 is based on Theorem 3 and on the properties of the group of P. S. Novikov with unsolvable conjugacy problem ([1]).

II. Let a finite alphabet $a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}$ and a finite number of pairs of words in this alphabet (A_i, B_i), $i = 1, 2, \dots, m$, be given.

Between the words of the given alphabet we introduce a binary relation: $X \leqslant Y$ if and only if this inequality is derivable by a finite number of applications of the following rules of inference: 1) $A_i \leqslant B_i$ $(i = 1, 2, \dots, m)$; 2) $a_j a_j^{-1} \leqslant 1$, $a_j^{-1} a_j \leqslant 1$, $1 \leqslant a_j a_j^{-1}$, $1 \leqslant a_j^{-1} a_j$ $(j = 1, 2, \dots, n)$; 3) $A \leqslant A$; 4) from $A \leqslant B$ and $B \leqslant C$ it follows that $A \leqslant C$; 5) from $A \leqslant B$ it follows that $XAY \leqslant XBY$, where A, B, X, and Y are arbitrary words.

We also define a relation of "equality" between words: 6) $A = B$ if and only if $A \leqslant B$ and $B \leqslant A$.

The algebraic system Γ so defined is, as is not hard to see, a partially ordered group. We shall say that the group Γ is **given by generators** $a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}$ **and defining inequalities** $A_i \leqslant B_i$ $(i = 1, 2, \dots, m)$.

When groups are given by generators and defining inequalities, it may happen that in the group Γ the reverse inequalities $B_i \leqslant A_i$ hold for all i. In this case the defining inequalities of the group Γ, according to rule 6), turn into equalities. Moreover, any group given by generators and defining equalities can in a trivial way be given by defining inequalities. Thus, specifying groups by generators and defining inequalities is a more universal way of specifying groups than specifying groups by means of generators and defining equalities.

For groups Γ, along with the algorithmic problems of identity and conjugacy, new algorithmic problems arise.

Word inequality problem: construct an algorithm which, for any pair of words A and B of the group Γ, determines whether the inequality $A \leqslant B$ holds.

Weak conjugacy problem: construct an algorithm which, for any pair of words A and B of the group Γ, determines whether there exists a word C such that the inequality $A \leqslant C^{-1}BC$ holds.

Theorem 6. *There exists a group Γ_1, given by a finite number of generators and a finite number of defining inequalities, for which the word inequality problem is unsolvable, while the identity problem is solvable.*

The construction of the group Γ_1 is based on the group of P. S. Novikov with unsolvable conjugacy problem ([1]).

We note that Theorem 6 strengthens Theorem 5, proved by other methods.

Theorem 7. *There exists a group Γ_2, with a finite number of generators and a finite number of defining inequalities, for which the weak conjugacy problem is unsolvable, while the word inequality problem and the conjugacy problem are solvable.*

Consider a calculus I over the alphabet $a_1, a_2, \dots, a_n$, in which word transformations are carried out according to the scheme: $X \to X$, $XA_{iY} \to XB_{iY}$, $aX \to Xa$, for arbitrary words X, Y and a letter a. (We agree to regard the words A_i and B_i as nonempty.)

Lemma 1. *There exists a calculus I_2 for which the word derivability problem is unsolvable.*

From the calculus I we construct a group Γ. The group Γ is given by the alphabet $a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}$ and by the defining inequalities

$$\begin{aligned} A_i p &\leqslant p B_i, \qquad && i = 1, 2, \dots, m; \\ a_j p &\leqslant p a_j, \qquad && j = 1, 2, \dots, n. \end{aligned}$$

Theorem 7 follows from Lemma 1 and the following principal lemma.

Principal Lemma. *In order that a word X be weakly conjugate to a word Y in the group Γ, where X and Y are words consisting of positive letters of the alphabet $a_1, a_2, \dots, a_n$, it is necessary and sufficient that Y be derivable from X in the calculus Γ.*

From Theorems 6 and 7 there follows the scheme:

Weak conjugacy problem ⟶ Inequality problem ⟶ Identity problem

where the dependence between algorithmic problems indicated by an arrow means that from the solvability of the given algorithmic problem for an arbitrary group Γ there follows the solvability in this group of the algorithmic problem following from it, but not conversely.

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