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1963

SovietRxiv

Abstract

Full Text

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APPLICATION OF THE METHOD OF LEAST SQUARES TO EIGENVALUE PROBLEMS

(Presented by Academician G. I. Petrov on 25 I 1963)

Let it be required to find the eigenvalues of the equation

$$A(\lambda)\varphi = 0, \tag{1}$$

i.e., the values λ for which equation (1) has a nontrivial solution; $A(\lambda)$ is a linear operator depending on the complex parameter λ, with domain of definition D_A in a Hilbert space H_1 and range in a Hilbert space H_2. We shall assume that for each eigenvalue there exists a neighborhood containing no other eigenvalues, and that, if λ is not an eigenvalue, then it is regular, i.e., the operator $A^{-1}(\lambda)$, inverse to the operator $A(\lambda)$, is defined on all of H_2 and is bounded. Denote

$$\varkappa(\lambda) = \inf_{\varphi \in D_A} \frac{\|A(\lambda)\varphi\|^2_{H_2}}{\|\varphi\|^2_{H_1}}.$$

Obviously, $\varkappa(\lambda) \geq 0$. If $\lambda = \lambda_i$ is an eigenvalue, then $\varkappa(\lambda_i) = 0$. If λ is a regular value, then

$$\varkappa(\lambda) = \frac{1}{\|A^{-1}(\lambda)\|^2} > 0, \tag{2}$$

where $\|A^{-1}(\lambda)\|$ is the norm of the operator $A^{-1}(\lambda)$. Consequently, the characteristic equation for the eigenvalues will be $\varkappa(\lambda) = 0$.

Take a system of linearly independent vectors $\psi_k \in D_A$ $(k = 1, 2, ...)$, complete in the following sense: for arbitrary fixed λ, $\varphi \in D_A$, and $\varepsilon > 0$ there exists a linear combination $\sum_{k=1}^m a_k\psi_k$ such that

$$\left\|\varphi - \sum_{k=1}^m a_k\psi_k\right\|^2_{H_1} + \left\|A(\lambda)\left(\varphi - \sum_{k=1}^m a_k\psi_k\right)\right\|^2_{H_2} < \varepsilon.$$

A system complete in the indicated sense will henceforth be called A-complete. Denote

$$\varkappa_n(\lambda) = \min_{\{a_k\}} \frac{\left\|A(\lambda)\left(\sum_{k=1}^{n} a_k\psi_k\right)\right\|_{H_2}^2}{\left\|\sum_{k=1}^{n} a_k\psi_k\right\|_{H_1}^2}, \quad (n = 1, 2, ...).$$

Obviously, $\varkappa_n(\lambda) \geq \varkappa(\lambda)$. By virtue of the A-completeness of the system $\{\psi_k\}$,

$$\lim_{n\to\infty} \varkappa_n(\lambda) = \varkappa(\lambda).$$

In particular, $\varkappa_n(\lambda_i) \to 0$ as $n \to \infty$, if λ_i is some eigenvalue. Suppose now that the functions $\varkappa(\lambda)$, $\varkappa_n(\lambda)$ are continuous. Take a circle $|\lambda - \lambda_i| = \rho$ such that in the disk $|\lambda - \lambda_i| \leq \rho$ there are no eigenvalues other than $\lambda = \lambda_i$. Since $\lim_{n\to\infty} \varkappa_n(\lambda_i) = 0$, it follows that, beginning with some number, the inequality

$$\varkappa_n(\lambda_i) < \min_{|\lambda-\lambda_i|=\rho} \varkappa(\lambda),$$

will hold.

and, consequently, for such n the function $\varkappa_n(\lambda)$ has a local minimum at some point λ_{in},

$$|\lambda_{in} - \lambda_i| < \rho.$$

Since ρ is arbitrarily small, the following is true.

Theorem. *For any eigenvalue λ_i of equation (1) there exists a sequence of points λ_{in} at which the corresponding functions $\varkappa_n(\lambda)$ have a local minimum, $\lim\limits_{n\to\infty} \lambda_{in} = \lambda_i$, and moreover $\lim\limits_{n\to\infty} \varkappa_n(\lambda_{in}) = 0$.*

Thus, as approximate eigenvalues one must take those values of λ at which the function $\varkappa_n(\lambda)$ has a local minimum.

Let us note that if there exists a point λ_0 at which the function $\varkappa(\lambda)$ has a local minimum different from zero, then there exists a convergent sequence of points λ_{0n} at which the corresponding functions $\varkappa_n(\lambda)$ have a local minimum, although $\lambda_0 = \lim\limits_{n\to\infty} \lambda_{0n}$ is not an eigenvalue. The points λ_{0n} are extraneous solutions, which must be rejected. To do this, it is necessary to verify that $\lim\limits_{n\to\infty} \varkappa_n(\lambda_{0n}) \neq 0$. In practical application of the method such a check may prove impossible; therefore it is important to determine whether the function $\varkappa(\lambda)$ has local minima different from zero. Suppose that at regular points the operator $A^{-1}(\lambda)$ is a holomorphic function of λ, i.e., in some neighborhood of an arbitrary regular point λ_0 it can be represented by the norm-convergent series

$$A^{-1}(\lambda) = A^{-1}(\lambda_0) + (\lambda - \lambda_0)A_1 + (\lambda - \lambda_0)^2 A_2 + (\lambda - \lambda_0)^3 A_3 + \cdots.$$

Then the mean-value theorem holds,

$$A^{-1}(\lambda_0) = \frac{1}{2\pi\rho}\int_C A^{-1}(\lambda)\,ds,$$

where C is a circle of sufficiently small radius ρ with center at the point λ_0. Hence the inequality follows

$$\|A^{-1}(\lambda_0)\| \le \frac{1}{2\pi\rho}\int_C \|A^{-1}(\lambda)\|\,ds,$$

and, consequently, $\|A^{-1}(\lambda)\|$ cannot have a local maximum at the regular point λ_0. From (2) it follows that $\varkappa(\lambda)$ cannot have a local minimum different from zero.

The equation satisfied by $\varkappa_n(\lambda)$ has the form

$$|a_{ij} - \varkappa_n\beta_{ij}| = 0, \tag{3}$$

where $\varkappa_n(\lambda)$ is the smallest root of equation (3), with

$$a_{ij} = (A(\lambda)\psi_j, A(\lambda)\psi_i)_{H_2}, \qquad \beta_{ij} = (\psi_j, \psi_i)_{H_1} \quad (i, j = 1, 2, 3, \dots, n).$$

If $\{\psi_k\}$ is an orthonormal system in H_1, then equation (3) is simplified, and $\varkappa_n(\lambda)$ is the smallest characteristic number of a Hermitian matrix.

Let us dwell on the choice of an A-complete system. Suppose the operator $A(\lambda)$ can be represented in the form

$$A(\lambda) = T(\lambda)A_0,$$

where A_0 is a linear operator with domain coinciding with D_A, with values in H_2, and having a bounded inverse operator A_0^{-1}; the operator $T(\lambda)$, defined in H_2, is bounded. Take a system $\{f_k\}$, complete in H_2. Then the system $\psi_k = A_0^{-1}f_k$ $(k = 1, 2, \dots)$ will be A-complete. Indeed, let $\varphi \in D_A$. Then $f = A_0\varphi \in H_2$, and for arbitrary $\varepsilon > 0$ there will be

such a linear combination $\sum_{k=1}^m a_k f_k$ that

$$\left\| f - \sum_{k=1}^{m} a_k f_k \right\|_{H_2}^2 < \varepsilon.$$

Further,

$$\left\|\varphi - \sum_{k=1}^{m} a_k \psi_k\right\|_{H_1}^2 = \left\|A_0^{-1}\left(f - \sum_{k=1}^{m} a_k f_k\right)\right\|_{H_1}^2 < \left\|A_0^{-1}\right\|^2 \varepsilon,$$

$$\left\|A(\lambda)\left(\varphi - \sum_{k=1}^{m} a_k \psi_k\right)\right\|_{H_2}^2 = \left\|T(\lambda)\left(f - \sum_{k=1}^{m} a_k \psi_k\right)\right\|_{H_2}^2 < \left\|T(\lambda)\right\|^2 \varepsilon.$$

The last two inequalities prove, by virtue of the arbitrariness of ε, that the system $\{\psi_k\}$ is A-complete. If $H_1 = H_2$, and the system of eigenvectors of the equation

$$A_0\psi - \lambda\psi = 0$$

is complete in $H_1 = H_2$, then as an A-complete system one may take the system of eigenvectors of this equation. In some problems such a choice of the system may substantially facilitate the computation of the elements of the determinant in equation (3).

Let us consider in more detail the case when $H_1 = H_2$ and equation (1) has the form

$$A\varphi - \lambda\varphi = 0, \tag{4}$$

where A is a self-adjoint linear operator having a complete system of eigenvectors $\{\varphi_k\}$ in $H_1 = H_2$, with the corresponding system of eigenvalues $\{\lambda_k\}$. In this case, as is known, all λ_k are real, and the system $\{\varphi_k\}$ may be regarded as orthonormal. For arbitrary $\varphi \in D_A$ and real λ we obtain, using the closure equation,

$$\frac{\|A\varphi - \lambda\varphi\|^2}{\|\varphi\|^2} = \frac{\sum_{k=1}^{\infty} |(A\varphi - \lambda\varphi, \varphi_k)|^2}{\|\varphi\|^2} = \frac{\sum_{k=1}^{\infty} |(\varphi, A\varphi_k - \lambda\varphi_k)|^2}{\|\varphi\|^2} =$$

$$= \frac{\sum_{k=1}^{\infty} (\lambda_k - \lambda)^2 |(\varphi, \varphi_k)|^2}{\|\varphi\|^2} \geq (\lambda_i - \lambda)^2 \frac{\sum_{k=1}^{\infty} |(\varphi, \varphi_k)|^2}{\|\varphi\|^2} = (\lambda_i - \lambda)^2,$$

where λ_i is the eigenvalue nearest to λ. Putting $\varphi = \varphi_i$, we obtain

$$\frac{\|A\varphi_i - \lambda\varphi_i\|^2}{\|\varphi\|^2} = (\lambda_i - \lambda)^2,$$

and, consequently,

$$\varkappa(\lambda) = (\lambda_i - \lambda)^2. \tag{5}$$

The above reasoning repeats the reasoning from work ([2]) for a second-order differential equation. In contrast to work ([2]), we do not fix λ. Since $\varkappa_n(\lambda) \geq \varkappa(\lambda)$, from (5) we obtain

$$(\lambda_i - \lambda)^2 \leq \varkappa_n(\lambda) \quad \text{or} \quad |\lambda_i - \lambda| \leq \sqrt{\varkappa_n(\lambda)}.$$

In particular,

$$|\lambda_i - \lambda_{in}| \leq \sqrt{\varkappa_n(\lambda_{in})}, \tag{6}$$

where λ_{in} is the point at which $\varkappa_n(\lambda)$ has a local minimum. Since

$$\lim_{n\to\infty} \varkappa_n(\lambda_{in}) = 0,$$

simultaneously with an approximate eigenvalue we obtain from formula (6) an estimate of the accuracy.

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Received
24 I 1963

CITED LITERATURE

1. S. G. Mikhlin, *Direct Methods in Mathematical Physics*, Moscow–Leningrad, 1950.

2. N. M. Krylov, N. N. Bogolyubov, *Izv. AN SSSR, OMEN*, No. , 471 (1929).

Note: Figure translations are in progress. See original paper for figures.