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Abstract

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A PROBLEM IN THE SPECTRAL THEORY OF DIFFERENTIAL OPERATORS

(Presented by Academician I. M. Vinogradov on 25 IV 1963)

Let D be the square in the plane (x, y): $0 \le x \le 1,\ 0 \le y \le 1$; let B be its boundary. Consider the boundary-value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0, \qquad (x, y) \in D, \qquad u(x, y)\big|_B = 0.$$

The eigenvalues of this problem are the numbers $\lambda = \pi^2(n^2 + m^2)$, where $n \ge 1,\ m \ge 1$ are integers. The multiplicity of the eigenvalue λ is equal to the number of representations of the number λ in the indicated form. The normalized eigenfunctions corresponding to this value of λ are

$$\omega(x, y) = 4 \sin^2 n\pi x \, \sin^2 m\pi y.$$

One of the well-known problems in the theory of differential operators is the study of the asymptotic properties (as $T \to \infty$) of the function $\Phi(T)$, representing the number of eigenvalues not exceeding T. Each eigenvalue is counted as many times as its multiplicity. The value of $\Phi(T)$ is equal to the number of lattice points in the region $x^2 + y^2 \le T/\pi^2$, $x \ge 1,\ y \ge 1$.

With the aid of the estimate of En Wen-lin [1] we obtain the number of lattice points in a circle

$$\Phi(T) = \sum_{\substack{n^2+m^2 \le T/\pi^2 \\ n \ge 1,\ m \ge 1}} 1 = \frac{T}{4\pi} - \frac{\sqrt{T}}{\pi} + O\big(T^{12/37+\varepsilon}\big)\,, \tag{1}$$

$\varepsilon > 0$. Closely connected with this problem is the problem of the asymptotic behavior, as $T \to \infty$, of the quantity

$$4 \sum_{\substack{n^2+m^2 \le T/\pi^2 \\ n \ge 1,\ m \ge 1}} \sin^2 n\pi x \, \sin^2 m\pi y,$$

where (x, y) is a fixed interior point of the square D. Denote by $l(x, y)$ the distance from the point (x, y) to the boundary of the square. The following result is obtained without difficulty:

Theorem 1. As $T \to \infty$,

$$4 \sum_{\substack{n^2+m^2 \leq T/\pi^2 \\ n \geq 1,\ m \geq 1}} \sin^2 n\pi x \, \sin^2 m\pi y = \frac{T}{4\pi} + O\left(\frac{\sqrt{T}}{l(x, y)}\right). \tag{2}$$

Using the method for estimating trigonometric sums of I. M. Vinogradov [2], this theorem can be strengthened.

Theorem 2. As $T \to \infty$,

$$4 \sum_{\substack{n^2+m^2 \leq T/\pi^2 \\ n \geq 1,\ m \geq 1}} \sin^2 n\pi x \, \sin^2 m\pi y =$$

$$= \frac{T}{4\pi} + O\left(\frac{T^{1/3} \ln T}{l(x, y)}\right) + O\left(\frac{1}{(l(x, y))^2}\right) + O\left(\frac{T^{1/4}}{(l(x, y))^{3/2}}\right). \tag{3}$$

Apparently, by carrying out the calculation more economically, the exponent 1/3 in estimate (3) can be somewhat lowered.

In equalities (2), (3), the constants in the O-symbols are absolute.

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CITED LITERATURE

[1] Yin Wen-lin, *Sci. Sinica*, **11**, No. 1, 10 (1962). [2] I. M. Vinogradov, *Selected Works*, Moscow, 1952.

Note: Figure translations are in progress. See original paper for figures.