TWO TYPES OF TERMINATION OF TRAJECTORIES OF A MARKOV PROCESS

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Abstract

Full Text

TWO TYPES OF TERMINATION OF TRAJECTORIES OF A MARKOV PROCESS

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In the modern theory of Markov processes one considers processes that are terminated at a random time ζ. In this note a class of σ-processes is singled out which, roughly speaking, are characterized by the fact that for them one may speak of a definite point of termination $x_{\zeta-0}$. In a certain sense the opposite class is the class of δ-processes. Under certain restrictions a general process is constructed from uniquely determined σ- and δ-processes. In accordance with ([1]) we introduce the notation: $(E, \mathcal{B})$ is the phase space (measurable space); Ω is the space of elementary events; $x_t = x_t(\omega)$ is the trajectory of the process; $\mathcal{N}$ is the σ-algebra in Ω generated by the sets $\{x_t \in \Gamma\}$ $(t \geq 0,\ \Gamma \in \mathcal{B})$; $P(t, x, dy)$ is the transition function; P_x are probability measures on $\mathcal{N}$. We also introduce the notation:

$$\xi(\Gamma) = \inf\{t:\ x_t \in \Gamma\} \quad (\Gamma \in \mathcal{B});$$

$$f(x_{\zeta-0}) = \lim_{t\to\zeta-0} f(x_t)$$

(if the limit exists); $\psi_s(\omega)$ is the realization coinciding with $x_t(\omega)$ for $t < s$ and terminating at the time s. Unspecified notation corresponds to ([1]). All processes under consideration are assumed to be standard (see ([2])). Processes are considered up to equivalence.

1. Put

$$\Gamma_n = \left\{x:\ P\left(\frac{1}{n}, x, E\right) > \frac{1}{n}\right\}.$$

Then $\xi(\Gamma_n) \uparrow \zeta$ (a.s. P_x). Terminating trajectories that are bounded by one of the sets Γ_n, i.e. $\xi(\Gamma_n) = \zeta$, will be called σ-trajectories. Nonterminating trajectories and trajectories for which $\xi(\Gamma_n) < \zeta$ $(n = 1, 2, \ldots)$, $\xi(\Gamma_n) \uparrow \zeta$, will be called δ-trajectories. If $\{\tau_n < t < \zeta\} \in \mathcal{N}_t$ $(n = 1, 2, \ldots)$ and $\tau_n \uparrow \zeta$, then on almost all (a.s.) σ-trajectories, starting with some n, $\tau_n = \zeta$. On a.s. δ-trajectories the quantities $\tau_n = \min\{n, \xi(\Gamma_n)\}$ do not coincide with ζ for any n. If E is a complete metric space and X has no discontinuities of the second kind, then for a.s. σ-trajectories there exists $x_{\zeta-0}$. If the semigroup T_t maps into itself the space of continuous functions tending to 0 at infinity, then conversely: a.s. trajectories for which $x_{\zeta-0}$ exists are σ-trajectories. Introduce the notation: Ω^σ_X is the set of σ-trajectories of the process X; Ω^δ_X is the set of δ-trajectories;

$$p_\sigma(x) = P_x(\Omega^\sigma_X), \qquad p_\delta(x) = P_x(\Omega^\delta_X).$$

Let

$$e(x) = M_x \min\{\zeta, 1\}.$$

Then

$$\Omega_X^\sigma = \left\{0 < \inf_{t<\zeta} e(x_t) \leq \sup_{t<\zeta} e(x_t) < 1\right\};$$

$$\Omega_X^\delta \cap \{\zeta < \infty\} = \{e(x_{\zeta-0}) = 0\};$$

$$\{\zeta = \infty\} = \{e(x_{\zeta-0}) = 1\} \quad \text{(a.s. } P_x).$$

2. We shall call a process a σ-process if $p_\sigma(x) \equiv 1$, and a δ-process if $p_\delta(x) \equiv 1$. Introduce the notation:

$$\bar{a}A = \{\psi_s\omega : \ \omega \in A, \ s \leq \zeta(\omega)\},$$

where $A \subseteq \Omega$. A property of a σ-process: if $A \subseteq \Omega$, $\overline{P}_x(A) = 1$, then $\overline{P}_x(\bar{a}A) = 1$. For a δ-process, on the contrary, if $\overline{P}_x(A) = 1$, then $\overline{P}_x(\bar{a}A) = 0$. The class of δ-processes is characterized by the fact that there exists $\{\Gamma_n\}$ ($\Gamma_n \in \mathcal{B}$) such that for a.s. $\omega \in \{\zeta < \infty\}$

$$\xi(\Gamma_n) < \zeta, \qquad \xi(\Gamma_n) \uparrow \zeta.$$

If for a δ-process $P(t, x, E) < 1$ ($t > 0, \ x \in E$), then for

$$\Gamma_n = \left\{x : \ \frac{1}{n} < P\left(\frac{1}{n}, x, E\right) < 1 - \frac{1}{n}\right\}$$

we have $\xi(\Gamma_n) < \zeta$,

$\xi(\Gamma_n) \uparrow \zeta$ (a.s. P_x). The σ-processes include, for example, Feller processes in a compact space E with $P_x\{\zeta < \infty\} \equiv 1$. For a σ-process without ruptures of the second kind in a complete metric space, a.s. P_x there exists $x_{\zeta-0}$.

3. For what follows we shall need one type of transformation of a process. Let $f(x)$ be an excessive function (e.f.) of the process X with transition function $P(t, x, dy)$ (see (2)). Denote by X^f the process in the phase space $\{x : f(x) > 0\}$, corresponding to the transition function

$$P'(t, x, dy) = P(t, x, dy)[f(x)]^{-1} f(y).$$

We denote the corresponding measures on $\mathcal{N}$ by P_x^f. For $A \in \mathcal{N}^t$,

$$P_x^f(A) = [f(x)]^{-1} M_x[\chi_A(\omega) f(x_t)].$$

If $\{\tau < t < \zeta\} \in \mathcal{N}^t$, $\tau = \zeta$ (a.s. P_x), then $\tau = \zeta$ (a.s. P_x^f). Hence it follows that if X, a.s., has one of the following properties: a) it is continuous, b) it has no ruptures of the second kind, c) it is F-bounded, where $F = \{\Gamma_n\}$, $\Gamma_n \in \mathcal{B}$ ($n = 1, 2, ...$), then, a.s., this property holds for X^f.

4. Let X_1, X_2 be two processes. We put $X \in S\{X_1, X_2\}$ if there exist e.f.' s $f_1(x), f_2(x)$ for X such that $X_1 = X^{f_1}$, $X_2 = X^{f_2}$, $f_1(x) + f_2(x) \equiv 1$. Let X_1, X_2 be processes in phase spaces E_1, E_2 with transition functions $P_1(t, x, dy), P_2(t, x, dy)$. Then, in order that $S(X_1, X_2)$ be nonempty, it is necessary and sufficient that: a) the set $E_1 \setminus E_2$ be absorbing for X_1, and $E_2 \setminus E_1$ for X_2; b) for $x \in E_1 \cap E_2$, $dy \subset E_1 \cap E_2$,

$$P_2(t, x, dy) = P_1(t, x, dy)[f(x)]^{-1} f(y),$$

where $f(x)$ is an e.f. for X_1, $f(x) > 0$ for $x \in E_1 \cap E_2$. If $X_0 \in S(X_1, X_2)$, then $S(X_1, X_2)$ coincides with the set of processes $X_0^{f_1 + cf_2}$, where $c > 0$. If $X_0 \in S(X_1, X_2)$, P_{0x}, P_{1x}, P_{2x} are the corresponding measures on $\mathcal{N}$, and $f_1(x), f_2(x)$ are e.f.' s for X_0 such that $X_0^{f_1} = X_1$, $X_0^{f_2} = X_2$, then for $A \in \mathcal{N}$

$$P_{0x}(A) = f_1(x) P_{1x}(A) + f_2(x) P_{2x}(A).$$

It follows from this that if $\overline{P}_{1x}(A) = 1$, $\overline{P}_{2x}(B) = 1$, then

$$\overline{P}_{0x}(A \cup B) = 1.$$

5. Let X be some process. Then

$$X \in S(X^{p_\sigma}, X^{p_\delta}), \qquad P_x^{p_\sigma}(\Omega_x^\sigma) = 1, \qquad P_x^{p_\delta}(\Omega_x^\delta) = 1.$$

We shall call X^{p_σ} the σ-component, X^{p_δ} the δ-component. If $p_\sigma(x) > 0$ $(x \in E)$, then the σ-component is a σ-process; the δ-component is always a δ-process. Thus, if $p_\sigma(x) > 0$ $(x \in E)$, then $X \in S(X_1, X_2)$, where X_1 is a σ-process and X_2 is a δ-process. If $p_\delta(x) > 0$ $(x \in E)$, then X_1, X_2 are determined uniquely.

6. We shall call an e.f. $f(x)$ regular if

$$P_x^f\{f(x_{\zeta-0}) < \infty\} = 1,$$

and singular if

$$P_x^f\{f(x_{\zeta-0}) = \infty\} = 1.$$

Introduce the notation

$$dA = \{\psi_s \omega : \omega_f \in A,\ s \leq \zeta(\omega)\} \quad (A \subset \Omega).$$

If $f(x)$ is regular, $A \in \mathcal{N}$, $\overline{P}_x(A) = 1$, then

$$\overline{P}_x(dA) = 1.$$

If $f(x)$ is singular, $A = \{f(x_{\zeta-0}) < \infty\}$, then $\overline{P}_x(A) = 1$,

$$\overline{P}_x(dA) = 0.$$

An e.f. $f(x)$ decomposes into the sum of a regular $f_1(x)$ and a singular $f_2(x)$, and in a unique way:

$$f_1(x) = f(x) P_x^f \{f(x_{\zeta-0}) < \infty\},$$

$$f_2(x) = f(x) P_x^f \{f(x_{\zeta-0}) = \infty\}.$$

When e.f.' s are added, their components are added. E. B. Dynkin proved that $f(x)$ is regular if $f(x) = M_x \xi$, where ξ is an excessive random variable (see (3)).

7. Let $f(x)$ be an e.f. for X; $f(x) > 0$ $(x \in E)$, $T_\infty f(x) \equiv 0$; let $f_1(x), f_2(x)$ be the regular and singular components of $f(x)$; $Y = X^f$, and Y_1, Y_2 the σ-component and δ-component of Y. Then, if

$$p_\delta(x) \equiv P_x\{\zeta = \infty\},$$

then $f_1(x)$ maps X to Y_1, and $f_2(x)$ maps X to Y_2.

Remark. The condition

$$p_\delta(x) \equiv P_x\{\zeta = \infty\}$$

means that, a.s., the δ-trajectories are nonterminating, and is fulfilled for nonterminating processes and σ-processes.

8. If $f(x)$ is a positive e.f. for X, then the corresponding transformation is invertible: if $Y = X^f$, then $X = Y^{1/f}$. Consider positive e.f.' s that transform X into a σ-process or a δ-process. Let $P_x\{\zeta < \infty\} \equiv 1$. Put

$$e(x) = M_x \min\{\zeta, 1\}.$$

Then $e(x)$ is an e.f., $e(x) > 0$ $(x \in E)$;

$Y = X^\varepsilon$ is a σ-process; $X = Y^f$, where $f(x) = \dfrac{1}{e(x)}$. Consider processes for which one of the functions $p_\sigma(x)$, $p_\delta(x)$, $r(x) = P_x\{\zeta = \infty\}$ is positive.

1) If $p_\sigma(x) > 0$ $(x \in E)$, then $X = Y^{1+f}$, where Y is a σ-process, $f(x)$ is a singular function for Y, and this representation is unique,

$$Y = X^{p_\sigma}, \qquad f(x) = \frac{1 - p_\sigma(x)}{p_\sigma(x)}.$$

2) If $r(x) > 0$ $(x \in E)$, then $X = Y^{1+f}$, where Y is a nonterminating process, $f(x)$ is an excessive function for Y, $T'_\infty f(x) \equiv 0$, where T'_t is the semigroup for Y. The representation is unique,

$$Y = X^r, \qquad f(x) = \frac{1 - r(x)}{r(x)}.$$

3) If $p_\delta(x) > 0$ $(x \in E)$, then $X = Y^f$, where Y is a δ-process, for example,

$$Y = X^{p_\delta}, \qquad f(x) = \frac{1}{p_\delta(x)}.$$

9. Every continuous process X is a subprocess of a continuous δ-process (a generalization of Theorem 1 ([4])). Consider the relation between trajectories of the two types for the process X and the subprocess Y corresponding to the multiplicative functional $\alpha_t > 0$ $(t > \zeta)$ (see (1)). We have $\Omega_Y^\delta = \Omega_X^\delta \cap \{\alpha_{\zeta-0} > 0\}$. It follows that, if $\alpha_{\zeta-0} = 0$ (a.s. P_x) or X is a σ-process, then Y is a σ-process.

10. **Some formulas.**

$$p_\sigma(x) = \lim_{n\to\infty} \lim_{h\to 0} h^{-1} \int_0^\infty \{T_t[\chi_n(x)\,[1 - P(h, x, E)]]\}\,dt,$$

where $\chi_n(x)$ is the indicator of the set

$$\left\{x: \; P\left(\frac{1}{n}, x, E\right) \geq \frac{1}{n}\right\}.$$

The regular component $f_1(x)$ of the excessive function $f(x)$ is given by the formula

$$f_1(x) = \lim_{n\to\infty} \lim_{h\to 0} h^{-1} \int_0^\infty \{T_t[\chi_{\{f(x)<n\}}(x)\,[f(x) - T_h f(x)]]\}\,dt,$$

if $T_\infty f(x) \equiv 0$, and by the formula

$$f_1(x) = \lim_{n\to\infty} T_\infty[\chi_{\{f(x)<n\}}(x)\,f(x)].$$

if $T_\infty f(x) \equiv f(x)$. The proof of the formulas is based on Lemma 1.1 ([5]).

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Note: Figure translations are in progress. See original paper for figures.