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Abstract

Full Text

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Semiheaps with Minimal Left Ideals

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A **semiheap** (1) is a set S on which a ternary algebraic operation is defined, assigning to each triple of elements $s_1, s_2, s_3 \in S$ an element $[s_1, s_2, s_3] \in S$ and satisfying the following associativity-type condition*:

$$\forall_{s_i \in S}\ [[s_1 s_2 s_3]\, s_4 s_5] = [s_1\, [s_4 s_3 s_2]\, s_5] = [s_1 s_2\, [s_3 s_4 s_5]].$$

The present note is devoted to one of the far-reaching analogies between the theory of semigroups and the theory of semiheaps—the transfer to semiheaps of the theory of completely simple semigroups of Sushkevich–Rees–Clifford (2,3,11).

1. A nonempty subset A of a semiheap S is called its **left** (respectively, **right**, **lateral**) **ideal** if

$$[SSA] \subseteq A \qquad (\text{respectively } [ASS] \subseteq A, \quad [SAS] \subseteq A).$$

If a subset $A \subseteq S$ is simultaneously a left, right, and lateral ideal, then it is called simply an **ideal** of the semiheap S. If a semiheap S contains an element 0 such that $[SS0] = [S0S] = [0SS] = 0$, then 0 is called the **zero** of the semiheap S. A left (right) ideal A of the semiheap S is called **minimal** if it is distinct from 0 and contains no proper (i.e., distinct from A and 0) left (right) ideals of the semiheap S.

2. Let A be a group with an adjoined zero o, i.e. $A = G \cup \{o\}$, where G is a group, o is the zero of the semiheap A, and ψ is an inverse automorphism of the semiheap A such that, for some $s \in G$,

$$\forall_{x \in A}\ \psi\psi x = sxs^{-1}.$$

If $s = e$, then ψ is called an **involution** (1) of the semiheap A. In particular, A admits the **canonical** involution ξ: $\xi o = o,\ \xi a = a^{-1}$ for every $a \in G$.

Further, let I, Λ be two arbitrary sets, and let $P = (p_{\lambda\chi})$ and $Q = (q_{ij})$ be a ΛI-matrix (3,5) (respectively, an $I\Lambda$-matrix) over the semiheap A such that in each of their rows there exists at least one element distinct from zero and

$$\forall_{\chi,\lambda \in \Lambda} p_{\lambda\chi} = s^{-1} \cdot \psi p_{\chi\lambda}, \qquad \forall_{i,j \in I} q_{ij} = \Psi q_{ji} \cdot s.$$

Denote by S the set of all triples $(a, i\lambda)$, where $a \in A,\ i \in I,\ \lambda \in \Lambda$, and regard all triples $(o, i\lambda)$ as equal to one another (denoting them by 0). Introduce in S the following ternary algebraic operation:

$$[(a, i\chi)(b, j\lambda)(c, k\mu)] = (ap_{\chi\lambda} \cdot \psi b \cdot q_{jk}c,\ i\mu). \tag{1}$$

With respect to this operation the set S, as is not difficult to verify, is a semiheap with zero 0. S contains no proper ideals, and is the union of its minimal left ideals

$$L_\lambda = \bigcup_{i \in I} (A, i\lambda)$$

and of its minimal right ideals

$$R_i = \bigcup_{\lambda \in \Lambda} (A, i\lambda)$$

(here $(A, i\lambda)$ denotes the set of all triples $(a, i\lambda)$ for fixed i, λ). We shall denote such a semiheap by $S(A, \psi, P_\Lambda, Q_I)$.

3. Suppose that under the conditions of item 2, I (respectively Λ) is the union of its disjoint nonempty subsets I_1, I_2 (respectively Λ_1,

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* For notation, see the article ([4]).

Λ_2); ψ is the canonical involution of the semigroup A. Further, let $p_{\chi\lambda} = o$ for $(\chi, \lambda) \in \Lambda_k \times \Lambda_k$ and $q_{ij} = o$ for $(i, j) \in I_k \times I_k$ $(k = 1, 2)$. Denote

$$T_k = \bigcup_{i \in I_k,\ \lambda \in \Lambda_k} (A, i\lambda) \quad (k = 1, 2).$$

The set $T = T_1 \cup T_2$, with respect to operation (1), is also a semigroup without proper ideals. As in Sec. 2, T coincides with the union of its minimal left ideals and with the union of its minimal right ideals. In contrast to the semigroup of Sec. 2, T is the union of its proper two-sided (i.e., left and right, but not necessarily lateral) ideals T_k, and moreover $T_k^{[3]} = [T_{kT_{kT}}k] = 0$.

4. A semigroup S is called i-simple if it contains no proper ideals and is distinct from the semigroup $V = \{0, a\}$, where 0 is a zero and $a^{[3]} = 0$. If the semigroup S contains at least one minimal left ideal, then denote by K the union of all minimal left ideals of the semigroup S.

Theorem 1. *If a semigroup S contains a minimal left ideal and $K^{[3]} \neq 0$, then S also contains a minimal right ideal.*

Theorem 2. *If an i-simple semigroup with zero S contains a minimal left or right ideal and $K^{[3]} \neq 0$, then S is isomorphic to one of the semigroups of Secs. 2–3.*

Theorem 3. *If an i-simple semigroup S with zero contains a minimal left ideal and contains no nilpotent left ideals or proper two-sided ideals, then it is isomorphic to one of the semigroups $S(A, \psi, P_\Lambda, Q_I)$.*

Theorem 4. *If an i-simple semigroup S with zero satisfies the minimality condition for left ideals (in particular, if S is finite), then it is isomorphic to one of the semigroups of Secs. 2–3.*

Theorem 5. *Every i-simple semigroup without zero, containing a minimal left or right ideal, is isomorphic to the subsemigroup of all nonzero elements of some semigroup $S(A, \psi, P_\Lambda, Q_I)$ (in which the matrices P and Q contain no zero elements).*

5. Let $S = S(A, \psi, P_\Lambda, Q_I)$, $S' = S(A', \psi', P'_{\Lambda'}, Q'_{I'})$, let c be an arbitrary element of $G' = A' \setminus \{0\}$, let φ be an isomorphism of the semigroup A onto A', let χ and ξ be one-to-one maps of the set I onto I' and of Λ onto Λ', respectively, and let r_i and s_χ be arbitrary elements of G, with $i' = \chi i \in I'$, $\lambda' = \xi\lambda \in \Lambda'$, $z \in A'$. For any such i, λ,

$$p'_{\chi'\lambda'} = s_\chi \cdot \varphi p_{\chi\lambda} \cdot c \cdot \psi' s_\lambda, \qquad q'_{i'j'} = \psi' r_i \cdot c^{-1} \cdot \varphi q_{ij} \cdot r_j, \qquad \psi' z = c^{-1} \cdot \varphi\psi\varphi^{-1} z \cdot c.$$

By a direct check one can verify that the map

$$f(x, i\chi) = (r_i^{-1} \cdot \varphi x \cdot s_\chi^{-1},\ i'\chi')$$

is an isomorphism of the semigroup S onto S'. Denote by Φ the class of all such isomorphisms.

Theorem 6. *Every isomorphism of the semigroup $S(A, \psi, P_\Lambda, Q_I)$ is contained in the class Φ.*

This theorem is analogous to the known description of isomorphisms of completely simple semigroups $(3, 5)$.

A similar mapping of the semigroup T of Sec. 3 is also its isomorphism if $\chi I_k = I'_k$, $\xi\Lambda_k = \Lambda'_k$ $(k = 1, 2)$, $c = e$ (e is the identity of the semigroup A'); moreover, all isomorphisms of the semigroup T are exhausted by such mappings.

6. If a semigroup without proper ideals contains a minimal left (right) ideal and at least one nonnilpotent element, then it cannot be isomorphic to the semigroup T of Sec. 3 or to the semigroup V of Sec. 4.

Theorem 7. Let S be a semiheap without proper ideals, containing a minimal left ideal. If S contains at least one non-idempotent element (in particular, if S does not contain a zero), then, under an isomorphism of the semiheap S (or of the semiheap $S \cup \{0\}$) onto the semiheap $S(A, \psi, P_\Lambda, Q_I)$, as the inverse automorphism ψ one can choose a certain involution of the semigroup A (see § 2).

In particular, if the semigroup S contains no left and right ideals distinct from S, then its elements form a group G with respect to a certain binary operation, and the operation in S is determined as follows:

$$\forall_{a,b,c \in S}\ [abc] = a \cdot p \cdot \psi b \cdot c, \tag{2}$$

where $p \in G$, and ψ is an involution of the group G. From this one can easily obtain the structure of an arbitrary heap ([6], [1]).

7. A semiheap S is called a **generalized heap** ([1]) if

$$\forall_{s \in S}\ [sss] = s,$$

$$\forall_{a,b,c \in S} [abbcc] = [accbb] \wedge [bbcca] = [ccbba] * .$$

Denote by U_X the XX-matrix over the semigroup A in which $u_{ii} = e$ and $u_{ij} = 0$ for $i \neq j$. It is easy to verify that the semiheap $S(A, \xi, U_\Lambda, U_I)$, where ξ is the canonical involution of the semigroup A, is a generalized heap.

Theorem 8. Let S be a generalized heap with zero and without proper ideals. If S contains a minimal left or right ideal, then it is isomorphic to the semiheap $S(A, \xi, U_\Lambda, U_I)$.

The analogous semiheap without zero is a heap.

8. **Theorem 9.** Let S be an i-simple semiheap containing a minimal left ideal. If $K^{[3]} = 0$, then S is the union of its subsemiheaps K and B, $K \cap B = \{0\}$, and

$$K^{[3]} = B^{[3]} = [KKB] = [KBB] = [BKK] = [BBK] = \{0\},$$

$$\forall_{k \in K}\ [BkB] = B, \qquad \forall_{b \in B}\ [KbK] = K,$$

$$K \setminus \{0\} = \bigcup_{i \in I,\ \lambda \in \Lambda} A_{i\lambda},$$

where I, Λ are certain sets, all $A_{i\lambda}$ are nonempty, and for any $i, j \in I$, $\lambda, \mu \in \Lambda$, $a_{i\lambda} \in A_{i\lambda}$, $c_{j\mu} \in A_{j\mu}$, $b \in B$,

$$[a_{i\lambda} b c_{j\mu}] \neq 0 \to \forall_{k \in I} \, [A_{k\lambda} b c_{j\mu}] = A_{k\mu}.$$

9. For the semiheaps of §§ 2–3 there exists an isomorphic representation by means of matrices over groups, analogous to the representation of completely simple semigroups ([3], [5]). To each element $a = (a, i\lambda)$ from the semiheap S of § 2 or § 3, put in correspondence a ΛI-matrix $\bar{a} = (x_{i\lambda})$ such that $x_{i\lambda} = a$, $x_{j\varkappa} = 0$ for $j \neq i$ or $\varkappa \neq \lambda$. Denote by $\bar{a}^*$ the ΛI-matrix obtained from $\bar{a}$ by transposition and by replacing all elements $x_{j\varkappa}$ by $\psi x_{j\varkappa}$. If on the set $\bar{S}$ of all such matrices $\bar{a}$ we introduce the operation $[\bar{a}\bar{b}\bar{c}] = \bar{a} P \bar{b}^* Q \bar{c}$, then the mapping $a \to \bar{a}$ is an isomorphism from S onto $\bar{S}$.

10. Let $S = S(A, \psi, P_\Lambda, Q_I)$; let N be such a normal divisor of the group $G = A \setminus \{0\}$ that $\psi N = N$; $\bar{A} = (G/N) \cup \{0\}$; further, let $\bar{\psi}(aN) = \psi a \cdot N$ for any $a \in A$, and $\bar{P}_\Lambda = (p_{\varkappa\lambda} N)$, $\bar{Q}_I = (q_{ij} N)$. Partition the set Λ into pairwise disjoint subsets $\bar{\lambda}$ so that $p_{\varkappa\lambda} = p_{\varkappa\nu}$ for any $\lambda, \nu \in \bar{\lambda}$, and denote $\bar{p}_{\varkappa\bar{\lambda}} = p_{\varkappa\lambda}$, where $\varkappa \in \bar{\varkappa}, \lambda \in \bar{\lambda}$. Let $\bar{\Lambda}$ be the class of all

* $\wedge$ is conjunction: $\alpha \wedge \beta$ means "α and β".

of such subsets $\bar{\lambda}$, $P_{\bar{\Lambda}} = (p_{\bar{\chi}\lambda})$. We define the set $\bar{I}$ and the matrix $Q_{\bar{I}}$ analogously. The mappings $\varphi_N(a, i\chi) = (aN, i\chi)$, $\varphi_l(a, i\chi) = (a, i\bar{\chi})$ $(i \in \bar{i})$, $\varphi_r(a, i\chi) = (a, \bar{i}\chi)$ $(\chi \in \bar{\chi})$ are homomorphisms of the semigroup S onto the semigroup $S(\bar{A}, \psi, \bar{P}_\Lambda, \bar{Q}_I)$, $S(A, \psi, P_\Lambda, Q_{\bar{I}})$, or, respectively, $S(A, \psi, P_{\bar{\Lambda}}, Q_I)$. The analogous mappings of the semigroup of item 3 are also its homomorphisms.

Theorem 10. *Every homomorphism of the semigroup of item 2 or item 3 can be represented as a superposition of no more than three homomorphisms of the form $\varphi_N, \varphi_l, \varphi_r$ (cf. with ([7],[8])).*

11. It is not difficult to verify that, by virtue of Theorem 10, the following semigroups have no nontrivial homomorphisms: a) the semigroup V of item 4; b) any semigroup S of item 6 with action (2), for which the corresponding group G is simple; c) every semigroup of the type described in items 2, 3, for which the group $G = A \setminus \{o\}$ is a group with identity, and the matrices P and Q do not contain two identical rows. However, in contrast to ([9],[10]), items a)–c) give an exhaustive description of semigroups with minimal left ideals and without nontrivial homomorphisms only in the case of semigroups without zero: it is unclear whether, among such semigroups, there can exist a semigroup S of item 8.

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Note: Figure translations are in progress. See original paper for figures.