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Abstract

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ON THE CLASSIFICATION OF SIMPLE SYSTEMS OF PATHS ON THE COMPLETE PRETZEL OF GENUS 2

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This article is a continuation of the works ([1,2]). Here we shall prove that, for a large class of simple systems of paths on the surface of the complete pretzel V of genus 2, equivalence (under automorphisms of the fundamental group)* of systems induced in the fundamental group of the complete pretzel is not only necessary but also sufficient in order to transform one system of paths into another by a homeomorphism $V \to V$.

§ 1. Let T be a two-dimensional torus with one hole, whose boundary we denote by u. We call a **chord** every simple directed arc v on T having its boundary points on u. In addition, it is required that the arc v not divide T. Two chords v, w are called **parallel** if, together with two arcs u' and u'' on u, they form the boundary of a disk on T. If $vu'w^{-1}u''$ is the boundary of this disk, then v and w have the "same direction." A chord v and a simple closed path k on T are called parallel if they divide T into two parts, one of which is an annulus with boundaries k and vu', where u' is an arc on u.

Let $\mathfrak{f}$ be a system of chords on T which do not intersect one another. The following assertions are proved with the aid of the corresponding propositions on curves on a closed torus.

a) $\mathfrak{f}$ is divided into no more than three classes $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ of parallel chords ($\mathfrak{f}_3$ or $\mathfrak{f}_2 \cup \mathfrak{f}_3$ may be empty). In each class we give all chords the same direction.

b) The classes and their chords can be denoted by

$$\mathfrak{f}_1 = \{v_1, \dots, v_l\}, \quad \mathfrak{f}_2 = \{w_1, \dots, w_m\}, \quad \mathfrak{f}_3 = \{x_1, \dots, x_n\},$$

and one can find a point a on u and such an orientation of the path u that first all the initial points are reached, and then the terminal points of the chords, in the course of motion along u from a in the orientation taken. In this case the chords are encountered at the initial points in the order

$$v_1, \dots, v_l, w_1, \dots, w_m, x_1, \dots, x_n$$

and at the terminal points in the order

$$v_l, \dots, v_1, w_m, \dots, w_1, x_n, \dots, x_1.$$

c) Let $\mathfrak{f}'$ be another system of chords, and suppose that in its classes there also lie l, m, or n chords. Then there exists a homeomorphism $\eta : T \to T$ with $\eta\mathfrak{f} = \mathfrak{f}'$.

d) Let $\mathfrak{f}$ be a system of chords and simple closed paths, not contractible on T. If curves from $\mathfrak{f}$ do not intersect one another, then they are parallel to one another.

We take on T two simple closed loops s and t with the following properties: s and t intersect u only at the initial point t; $s \cap t$ consists of one point. We suppose that the initial point t is not a boundary point of any chord of $\mathfrak{f}$. To each chord $y \in \mathfrak{f}$ there correspond its algebraic intersection numbers with s and t; we write: $y \sim (a, \alpha)$. Parallel chords have identical numbers. Let

$$v_i \sim (a, \alpha), \quad w_i \sim (b, \beta), \quad x_i \sim (c, \gamma).$$

d) One has

$$\begin{vmatrix} a & \alpha \\ b & \beta \end{vmatrix} = \begin{vmatrix} b & \beta \\ c & \gamma \end{vmatrix} = \begin{vmatrix} a & \alpha \\ c & \gamma \end{vmatrix} = \pm 1.$$

Hence it follows that, for $n > 0$,

$$b = a + c \quad \text{and} \quad \beta = \alpha + \gamma.$$

* A complete solution, for example, in ([4]).

Let η be a homeomorphism $T \to T$ and $\mathfrak{f}' = \eta\mathfrak{f}$. We shall denote by primes the intersection numbers of the system $\mathfrak{f}'$ with s and t. In the commutative group $H_1(T)$ of pairs of algebraic intersection numbers, η induces an automorphism ξ given by the matrix

$$\begin{pmatrix} a & \alpha \\ b & \beta \end{pmatrix}^{-1} \begin{pmatrix} a' & \alpha' \\ b' & \beta' \end{pmatrix}.$$

§ 2. Let d be a disk with boundary u, and hence let $T \cup d$ be a torus. We regard $T \cup d$ as the boundary of a solid torus W and take s to be a meridian of the solid torus W. The automorphism ξ of the group $H_1(T)$ induces an automorphism of the fundamental group of W only in the case when

$$\begin{pmatrix} a & \alpha \\ b & \beta \end{pmatrix}^{-1} \begin{pmatrix} a' & \alpha' \\ b' & \beta' \end{pmatrix} = \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}.$$

Precisely in this case one can extend the homeomorphism η and obtain a homeomorphism $W \to W$. Then either $a' = a$, $b' = b$, $c' = c$ or $a' = -a$, $b' = -b$, $c' = -c$. If another system $\mathfrak{f}''$ of arcs has first intersection numbers a, b, c, and in $\mathfrak{f}''$ there are l arcs of the 1st class, m of the 2nd, and n of the 3rd class, then $\mathfrak{f}''$ and $\mathfrak{f}$ are homeomorphic on W (with $m > 0$). Indeed,

$$\begin{vmatrix} a & \alpha'' \\ b & \beta'' \end{vmatrix} = \pm 1,$$

$$\alpha'' = \pm\alpha + va \quad \text{and} \quad \beta'' = \pm\beta + vb.$$

Moreover,

$$\gamma' = \pm\gamma + vc.$$

Thus we obtain $\mathfrak{f}''$ from $\mathfrak{f}$ by a homeomorphism of the surface T, which is characterized by the matrix

$$\begin{pmatrix} \pm 1 & v \\ 0 & \pm 1 \end{pmatrix}$$

(see (1)).

§ 3. Let V be the full pretzel of genus 2, and let d be a disk splitting V into two solid tori W_l and W_r. The boundary u of the disk d is called the belt (Taillenschnitt). Let $\mathfrak{f}$ be a simple system of paths on the surface V. The belt u divides $\mathfrak{f}$ into two systems $\mathfrak{f}_l$ and $\mathfrak{f}_r$, consisting of arcs and simple closed paths on W_l and W_r. Arcs y_l and y_r are called connected if they have a common boundary point. For the moment we restrict ourselves to systems that contain no closed paths in $\mathfrak{f}_l$ and $\mathfrak{f}_r$. There exists a section $Z = \{f_l, f_r\}$ of the full pretzel, not intersecting d. Let $\{s_l, s_r\}$ be the trace of the section. As in § 2, take sections of the surfaces of both solid tori containing the meridians s_l and s_r. If every arc of $\mathfrak{f}_l$ and $\mathfrak{f}_r$ has algebraic intersection number with s_l and s_r, respectively, not equal to zero, and if all complete words* of the paths of $\mathfrak{f}$ are cyclically reduced, then we say that $\mathfrak{f}$ occupies normal position (with respect to u and Z). The complete words have the form

$$S_l^{a_1} S_r^{b_1} \cdots S_l^{a_n} S_r^{b_n}.$$

In what follows we call the parts $S_l^{a_i}$ (or $S_r^{b_i}$) powers, in which the generator S_l (or S_r) occurs in the complete words. If $\mathfrak{f}$ is in normal position, then each power of one generator in the complete words corresponds to one arc, and conversely. The question arises how many simple systems of paths in normal position have the same complete words. Let $\mathfrak{f}$ and $\mathfrak{h}$ be such systems. We shall further assume that there exist homeomorphisms $\eta_l : W_l \to W_l$ and $\eta_r : W_r \to W_r$ with the following properties (i denotes l or r):

e) $\eta_i d = d$.

) $\eta_i \mathfrak{f}_i = \mathfrak{h}_i$.

) η_l and η_r both preserve or both reverse orientation.

) At least two connected arcs are carried into a pair of connected arcs.

Then there exists a homeomorphism $\eta : V \to V$ with $\eta d = d$ and $\eta \mathfrak{f} = \mathfrak{h}$.

§ 4. Given $\mathfrak{f}$ and $\mathfrak{h}$ with identical complete words, homeomorphisms η_l and η_r satisfying e) and) exist in the case when the complete words "determine" how many elements are contained in each class of arcs—

* The complete word (Ablesung) of one path $\mathfrak{f}$ indicates, in the symbols S_l and S_r, in what order and from which directions $\mathfrak{f}$ pierces the disks of the section.

ties, and if there are at least two classes on both sides. Then it is possible to determine precisely the first number of intersections of the bonds of one class and to apply § 2. Conditions) and) are also satisfied, since the complete words express how many bonds of one class on W_l must be connected with bonds of one class on W_r at the initial (or terminal) point. It is easy to see that this can be done in only one way.

It remains to explain the expression "the complete words determine" : if the generator S_i occurs in the degree S_i^b ($|b| \geqslant 3$) in the complete words, then two bonds on W_i are parallel only in the case when they correspond to one and the same degree of the generator S_i. Hence, in one class the number of bonds is determined by the frequency with which this degree occurs in the complete words.

If the generator S_i occurs in the complete words only in the degrees $S_i^{\pm 2}$ and $S_i^{\pm 1}$, then the numbers of elements in the classes cannot be computed in the same way. The bonds may have, for example, the following numbers of intersections: $(1,0)$, $(2,1)$, $(1,1)$. It is necessary to calculate how many bonds with first number 1 lie in the first class, and how many in the third. This is often possible. Suppose, for example, that the complete words determine on W_r the classes of bonds, but do not determine them on W_l. If there are parts $(S_l^2 S_r^a)^{\pm 1}$ and $(S_l^2 S_r^b)^{\pm 1}$ with $a \neq b$, then one can compute how the bonds from f_l are divided into classes. The same is possible if the complete words contain only the degrees $S_l^{\pm 1}$, $S_l^{\pm 2}$, $S_r^{\pm 1}$, $S_r^{\pm 2}$, but there are parts $(S_l^2 S_r)^{\pm 1}$, $(S_l^2 S_r^{-1})$ and a pair $(S_r^2 S_l)^{\pm 1}$, $(S_r^2 S_l^{-1})^{\pm 1}$. In these cases the degrees S_l^{+1} (or S_l^{-1}) which follow the degrees S_r^q (or stand before S_r^{-q}) with $q > 0$, correspond to bonds of one class, while the others correspond to bonds of the other class.

§ 5. Let $\mathfrak{f}$ be any simple system of paths with induced system $\mathfrak{R}$ of elements of the fundamental group. Let $\mathfrak{R}'$ be an equivalent system*, consisting of cyclically reduced words. Then there exists a homeomorphism η such that $\mathfrak{R}'$ is the system of complete words with respect to Z for $\eta\mathfrak{f}$ (^{2}). Cut the surface of the complete pretzel by the cut Z and obtain a sphere with four holes f_l^{+1}, f_l^{-1}, f_r^{+1}, and f_r^{-1}. If each generator occurs in $\mathfrak{R}'$ at least once with an exponent different from ± 1, then, as in ([2]), one can construct a belt u' which does not intersect the arcs connecting f_i^{+1} and f_i^{-1}, and which pierces exactly once the arcs connecting f_l^{ε} with f_r^{δ}. There is a homeomorphism $\eta' : V \to V$ with $\eta' u' = u$, identical on Z. Then $\eta' \eta \mathfrak{f}$ is in normal position with respect to u and Z, and $\mathfrak{K}'$ is a system of complete words.

Thus, let $\mathfrak{f}$ and $\mathfrak{h}$ be simple systems of paths on the surface of a complete pretzel of genus 2 with induced systems $\mathfrak{K}$ and $\mathfrak{H}$ in the fundamental group. $\mathfrak{f}$ and $\mathfrak{h}$ are homeomorphic if there is a system $\mathfrak{S}$ equivalent to $\mathfrak{K}$ and $\mathfrak{H}$ with the following properties:

) $\mathfrak{S}$ is a system of cyclically reduced words.

) Each generator occurs in $\mathfrak{S}$ at least in two distinct and noninverse degrees.

) $\mathfrak{S}$ determines the classes of bonds.

This proposition generalizes the proposition on systems of paths lying in the homotopy classes $S_l^a S_r^b$ ([2]). Moreover, our normal form strengthens Dehn' s normal form ([3]).

§ 6. Especially important systems of paths are Heegaard splitting diagrams (in ([1]) *Diagramme normaler Zerlegungen*). We have proved that if a Heegaard splitting diagram of genus 2 has an induced system equivalent to a system $\mathfrak{S}$ with properties),), and), then its relations already determine the manifold represented in it.

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* Two systems $\{K_1, \dots, K_r\}$ and $\{K_1', \dots, K_r'\}$ are called **equivalent** if there exists an automorphism α such that $\alpha K_i = L_i K_i' L_i^{-1}$.

The following condition is also of interest:

′) every generator occurs in $\mathfrak{J}$ at least once with an exponent distinct from ± 1, and one generator occurs only in one power and in its inverse. If the induced system of a Heegaard splitting diagram is equivalent to a system $\mathfrak{J}$ satisfying) and ′), then the manifold represented is either a lens space, or the sum of two lens spaces (one of them may be a neighborhood of the sphere). This is also true if the induced system contains an element equivalent to S_i^a.

It follows from) that each generator occurs in the relations given by the Heegaard splitting diagram in normal position with no more than three positive exponents a, b, and c, and with exponents $-a$, $-b$, $-c$ (where $(a, b) = 1$ and $b = a + c$, if $c \neq 0$).

It can be proved that *every simple system of paths can be transformed into a system in normal position.* In particular, for diagrams of the sphere (and for manifolds admitting a Heegaard splitting of genus 2 and first Betti number 0) we even obtain an equivalent system whose system $\mathfrak{J}$ of full words satisfies) and). Unfortunately, it is not obvious that) is also satisfied. Let $\{K_1, K_2\}$ be the induced system. Take an equivalent system $\{H_1, H_2\}$ with minimal length in the generators. We exclude the case in which H_1 and H_2 both have length less than three; this is considered in ([2]). In the other case, in H_1 and H_2 there is

) a power S_l^a with $|a| > 1$

or

) a part $S_l^\varepsilon S_r^b S_l^\varepsilon$, where $\varepsilon = \pm 1$ and $b \neq 0$.

Otherwise the homology group of the manifold cannot be zero. If) is satisfied, but not), then let ξ be the automorphism defined by the mapping $S_l^\varepsilon \to S_l^\varepsilon S_r^{-b}$, $S_r \to S_r$. Then the cyclically reduced word for ξH_1 (or ξH_2) has a power S_l^c with $|c| \geq 2$. S_r also has an exponent distinct from ± 1. Otherwise, instead of some power $S_r^{b'}$ we would obtain only $S_r^{\pm 1}$, and the number of signs $S_l^{\pm 1}$ would not increase; thus the length would decrease, since S_r^b is canceled. This contradicts the minimality of the system $\{H_1, H_2\}$. If) is true, and S_r occurs only in the powers S_r and S_r^{-1} in H_1 and H_2, then there is a part $S_r^\varepsilon S_l^a S_r^\varepsilon$, where $\varepsilon = \pm 1$ and $a \neq 0$. Then define ξ by the mapping $S_r^\varepsilon \to S_r^\varepsilon S_l^{-a}$, $S_l \to S_l$, and in the cyclically reduced words of the elements ξH_1 and ξH_2 each generator has an exponent distinct from ± 1.

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Note: Figure translations are in progress. See original paper for figures.