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Abstract

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INCOMMENSURABILITY OF THE MINIMAL LINEAR MEASURE WITH THE LENGTH OF A SET

(Presented by Academician A. N. Kolmogorov on 14 II 1963)

In the work of A. N. Kolmogorov (1) the question is posed of the commensurability of the Hausdorff length of a set $l(M)$ with the minimal linear measure of the set $L(M)$, i.e., whether the inequality

$$l(M) \leq CL(M)$$

holds for every set M in the plane of finite positive length, where C is a positive absolute constant. In this note an example is constructed of a set M in the plane of finite positive length and with $L(M) = 0$.*

Notation and construction of the set. $d(M)$ is the diameter of the set M; $c(M)$ is the convex hull of the set M;

$$c(M) = \inf \sum_{\mu} d(S_\mu)$$

over all coverings $\{S_\mu\}$ of the set M by convex domains S_μ.

Further, let k be a positive integer; δ a positive number; τ a number equal to 0 or 1. $P_\tau(k, \delta, R)$ is a transformation which sends the horizontal segment $R = \{a \leq x \leq b,\ y = c\}$ into the set consisting of $2k$ horizontal segments of the form

$$\left\{a + \frac{(i-1)(b-a)}{k} \leq x \leq a + \frac{(2i-1)(b-a)}{2k}, \quad y = c + \delta\tau\right\},$$

where $i = 1, 2, \dots, k$.

If the set M consists of m horizontal segments r_l, then

$$P_\tau(k, \delta, M) = \bigcup_{l=1}^{m} P_\tau(k, \delta, r_l).$$

Let

$$M_0 = \{0 \le x \le 1,\ y = 0\},$$

$$M(\tau_1) = P_{\tau_1}(k_0, \delta_0, M_0),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$M(\tau_1, \dots, \tau_j) = P_{\tau_j}(k(\tau_1, \dots, \tau_{j-1}), \delta(\tau_1, \dots, \tau_{j-1}), M(\tau_1, \dots, \tau_{j-1})).$$

Thus the set $M(\tau_1, \dots, \tau_j)$ consists of

$$k_0 \cdot k(\tau_1) \cdots k(\tau_1, \dots, \tau_{j-1})$$

horizontal segments $r_l(\tau_1, \dots, \tau_j)$, lying on one straight line; all of them have the same length

$$l(r_l(\tau_1, \dots, \tau_j)) = \frac{1}{2^j k_0 \cdots k(\tau_1, \dots, \tau_{j-1})}.$$

Denote

$$M(\tau_1, \dots, \tau_j)^i = \bigcup_{\tau_{j+1}, \dots, \tau_{j+i} = 0;\,1} M(\tau_1, \dots, \tau_j, \tau_{j+1}, \dots, \tau_{j+i}).$$

Let $J_i(\tau_1, \dots, \tau_j)$ be the number of segments in the set $r_l(\tau_1, \dots, \tau_j)^i$

$$J_i(\tau_1, \dots, \tau_j) = 2^i \sum_{\tau_{j+1}, \dots, \tau_{j+i-1} = 0;\,1} k(\tau_1, \dots, \tau_j) \cdots k(\tau_1, \dots, \tau_{j+i-1}),$$

Now we construct the set M^{N_1}, having first imposed on the numbers

$$k_0, \dots, k(\tau_1, \dots, \tau_{N_1-1}), \quad \delta_0, \dots, \delta(\tau_1, \dots, \tau_{N_1-1}),$$

where $\tau_1, \dots, \tau_{N_1-1},\ \tau_{N_1} = 0;\ 1$, the following conditions:

a) $N_1 J_{N_1-1}(0) \le k(1)$,

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* A. N. Kolmogorov expressed the supposition of the equality of these measures: “Es scheint mir nicht unwahrscheinlich, daß immer $m_1(E) = \mu_1(E)$” (^{1}), p. 361).

. . . .

$$N_1 J_{N_1-j}(\tau_1, \dots, \tau_{j-1}, 0) \leq k(\tau_1, \dots, \tau_{j-1}, 1).$$

$$j = 1, \dots, N_1 - 1;$$

b) $N_1 \delta_0 \leq l(r_l(\tau_1))$,

. . . .

$$N_1 \delta(\tau_1, \dots, \tau_{j-1}) \leq l(r_l(\tau_1, \dots, \tau_j)).$$

$$j = 1, \dots, N_1;$$

c) $\delta_0 \geq 2^3 l(r_l(\tau_1, \tau_2))$,

. . . .

$$\delta(\tau_1, \dots, \tau_{j-1}) \geq 2^{j+2} l(r_l(\tau_1, \dots, \tau_j, \tau_{j+1}))$$

$$j = 1, \dots, N_1 - 1.$$

Thus, we have constructed the set M^{N_1}, consisting of a finite number J_{N_1} of intervals $r_l(\tau_1, \dots, \tau_{N_1})$. Now we construct the set $M^{N_1 N_2}$, applying to each interval in M^{N_1} the operation P_τ N_2 times with the same conditions a), b), only for the number N_2, while in condition c) we increase the exponents of the power of 2 by N_1: $\delta_0^1 \geq 2^{3+N_1} l(r_l(\tau_1, \tau_2))$. By $\delta^\nu(\tau_1, \dots, \tau_j)$, $k^\nu(\tau_1, \dots, \tau_j)$ are denoted the numbers δ, k for the $(\nu+1)$-st step of the construction. Further, applying to each interval of the set $M^{N_1 N_2}$ the operation P_τ N_3 times, with in condition c) the numbers $\delta^2(\tau_1, \dots, \tau_j)$ involving exponents of the form $3 + N_1 + N_2$, $4 + N_1 + N_2, \dots$, $N_3 + N_2 + N_1 + 1$, we construct the set $M^{N_1 N_2 N_3}$, and so on. We obtain the set $M^{N_1 \cdots N_k}$. Thus, let a sequence of natural numbers $N_1, \dots, N_k, \dots$ be given, the growth conditions on which will be imposed below; then

$$M = \lim_{k \to \infty} M^{N_1 \cdots N_k}.$$

Lemma 1. Let $\bar{d}(E) = \sup \max\{|x_2 - x_1|,\ |y_2 - y_1|\}$ over all $\{x_1, y_1\},\ \{x_2, y_2\} \in E$; S be a set. Then

$$\left(2 + \frac{1}{2^j} + \dots + \frac{1}{2^{j+i+1}}\right) \bar{d}(S) \geq l(S \cap M(\tau_1, \dots, \tau_j)^i);$$

$$j = 0, 1, \dots; \quad i = 0, 1, \dots .$$

Lemma 2. The coverings of the set M^{N_1} and of the sets $M^{N_1 \cdots N_n}$, where $n = 1, 2, \ldots,$

$$c(M^{N_1}) \geq \frac{1}{4\sqrt{2}}; \qquad c(M^{N_1 \cdots N_n}) \geq \frac{1}{4\sqrt{2}}.$$

This lemma follows at once from Lemma 1, taking into account that $d(S) \geq \frac{1}{\sqrt{2}} \bar{d}(S)$, since condition c) for $M^{N_1 \cdots N_n}$ coincides with condition c) for $M^{N_1 + \dots + N_n}$.

Lemma 3. The length of the set M

$$l(M) \geq \frac{1}{4\sqrt{2}}.$$

The lemma follows immediately from Lemma 2:

$$l(M) \geq c\left(\lim_{k \to \infty} M^{N_1, \dots, N_k}\right) \geq \lim_{k \to \infty} c\left(M^{N_1 \cdots N_k}\right) \geq \frac{1}{4\sqrt{2}}.$$

Lemma 4. For any contraction mapping f of the set M^i onto a straight line, the measure of the image

$$\operatorname{mes}(f(M^i)) \leq \frac{20}{i} l(M^i), \quad \text{where } i \leq N_1.$$

We prove this by induction on i. For $i \leq 20$ the assertion of the lemma is obvious. Suppose the lemma is true for all $M^i,\ i = 1, 2, \dots, n \leq N_1$.

We note that

$$M^{n+1} = M(0)^n \bigcup M(1)^n, \qquad f(M^{n+1}) = f(M(0)^n) \bigcup f(M(1)^n),$$

then

$$\operatorname{mes}(f(M^{n+1})) = \operatorname{mes}(f(M(0)^n)) + \operatorname{mes}(f(M(1)^n)) -$$

$$-\operatorname{mes}(f(M(0)^n)\cap f(M(1)^n)). \tag{1}$$

1. If

$$\operatorname{mes}(f(M(0)^n)) \le \frac{9}{10}\frac{20}{n}\, l(M(0)^n),$$

then, since

$$l(M(0)^n) = l(M(1)^n) = \frac{1}{2} l(M^{n+1}) = \frac{1}{2},$$

we obtain

$$\operatorname{mes}(f(M^{n+1})) \le \frac{9}{10}\frac{20}{n}\frac{1}{2} + \frac{20}{n}\frac{1}{2} \le \frac{20}{n+1}\, l(M^{n+1}),$$

since $n > 20$.

2. Let

$$\operatorname{mes}(f(M(0)^n)) > \frac{9}{10}\frac{20}{n}\, l(M(0)^n) = \frac{9}{n}.$$

The set $f(M(0)^n)$ on the line consists of no more than $k_0 J_n(0)$ intervals or points, since $f(M(0)^n)$ is a contracted image of the set $M(0)^n$, which consists of the same number of intervals.

Let

$$A_p = \inf(f(r_p(0)^n)), \qquad A'_p = \inf(f(r_p(1)^n)),$$

$$B_p = \sup(f(r_p(0)^n)), \qquad B'_p = \sup(f(r_p(1)^n));$$

then, since the distance from $r_p(0)^n$ to $r_p(1)^n$ is equal to δ_0,

$$|A'_p - A_p| \le \delta_0, \qquad |B'_p - B_p| \le \delta_0,$$

i.e.

$$\operatorname{mes}(f(r_p(0)^n) \cap A'_p B'_p) \ge \operatorname{mes}(f(r_p(0)^n)) - 2\delta_0. \tag{2}$$

Further,

$$f(M(1)^n) = \bigcup_{p=1}^{k_0} f(r_p(1)^n) = \bigcup_{p=1}^{k_0} \bigcup_{l=1}^{k(1)} f(r_{pl}(1,0)^{\,n-1} \cup r_{pl}(1,1)^{\,n-1}).$$

Since the distance between two neighboring subsets of the form $f(r_{pl}(1,0)^{\,n-1} \cup r_{pl}(1,1)^{\,n-1})$, of which the set $f(r_p(1)^n)$ consists, is not greater than $l(r_{pl}(1,0)) = l(M(1)^n)/2k_0k(1)$, it follows that into any interval $CD:\ A'_p \leq C \leq D \leq B'_p$ there fall no fewer than $|D-C|/2l(r_{pl}(1,0)) - 1$ of these subsets. It is easy to see that into two intersecting intervals $CD, EF:\ C < E < D < F$, $A'_{p_1} \leq C < D \leq B'_{p_1}$ and $A'_{p_2} \leq E < F \leq B'_{p_2}$, there fall no fewer than $|C-F|/2l(r_{pl}(1,0)) - 2$ of these subsets. Hence, taking account of (2) and of the fact that the set $f(r_p(0)^n)$ consists of no more than $J_n(0)$ intervals, we obtain that this set $f(r_p(0)^n)$ "covers" no fewer than

$$\frac{\operatorname{mes}(f(r_p(0)^n)) - 2\delta_0}{2l(r_{pl}(1,0))}$$

$$- J_n(0)$$

subsets of the form $f(r_{pl}(1,0)^{\,n-1} \cup r_{pl}(1,1)^{\,n-1})$. This means that all of $f(M(0)^n)$ "covers" no fewer than

$$\frac{\operatorname{mes}(f(M(0)^n)) - 2\delta_0 k_0}{2l(r_{pl}(1,0))} - k_0 J_n(0) \geq \frac{\frac{9}{10}\frac{20}{n}\, l(M(0)^n) - 2\delta_0 k_0}{2l(r_{pl}(1,0))} - k_0 J_n(0) \qquad (3)$$

of these subsets. Taking (1), (3) into account, and

$$\operatorname{mes}\left(f\left(r_{pl}(1,0)^{\,n-1} \cup r_{pl}(1,1)^{\,n-1}\right)\right) \leq \frac{2 \cdot 20}{n-1}\, l\left(r_{pl}(1,0)\right),$$

we have

$$\begin{aligned}
\operatorname{mes}\left(f\left(M^{n+1}\right)\right) \leq \frac{20}{n}\frac{1}{2} + \frac{20}{n-1} \cdot 2l\left(r_{pl}(1,0)\right) \times \\
\times \left[k_0 k(1) - \frac{\frac{9}{n}\, l\left(M(0)^n\right) - k_0\delta_0}{l\left(r_{pl}(1,0)\right)} + k_0 J_n(0)\right] \\
= \frac{20}{n}\frac{1}{2} + \frac{20}{n-1}\frac{1}{2} - \frac{20 \cdot 9}{(n-1)n} + \frac{40}{n-1}\delta_0 k_0 + \frac{40}{n-1}\, l\left(r_{pl}(1,0)\right) k_0 J_n(0).
\end{aligned}$$

From a) it follows that $l(r_{pl}(1,0)) k_0 J_n(0) \leq 1/N_1$; from b) it follows that $2\delta_0 k_0 \leq 1/N_1$, whence

$$\operatorname{mes}\left(f\left(M^{n+1}\right)\right) \leq \frac{10}{n} + \frac{10}{n-1} + \frac{20}{(n-1)N_1} + \frac{40}{(n-1)N_1} - \frac{180}{n(n-1)} \leq \frac{20}{n+1},$$

where $20 < n \leq N_1$, as was required.

We now return to the conditions on the growth of the sequence $N_1, N_2, \dots, N_k, \dots$. Let a sequence of positive $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k, \dots$, tending to zero, be given. It follows from Lemma 4 that if only $N_1 > 20 \cdot 4/\varepsilon_1$, then for any contracted mapping f onto a line $\operatorname{mes}(f(M^{N_1})) \leq \varepsilon_1/4$. Denote by $\overline{E}_{\sigma_1}$ the σ_1-extension of the set E. Since the set M^{N_1} consists of a finite number of intervals J_{N_1}, for $\sigma_1 > 0$ and $4J_{N_1}\sigma_1 < \varepsilon_1$ the measure of the σ_1-extension of the image of the set M^{N_1}

$$\operatorname{mes}\left(\overline{f\left(M^{N_1}\right)}_{\sigma_1}\right) < \varepsilon_1.$$

Next, choose

$$N_2 > \max\left\{\frac{J_{N_1} \cdot 80}{\varepsilon_2}, \frac{1}{2\sigma_1}\right\};$$

it is immediately clear that $\operatorname{mes}(f(M^{N_1 N_2})) < \varepsilon_2/4$ for any contracted mapping f.

Find $\sigma_2 < \frac{1}{4}\sigma_1$ and

$$\operatorname{mes}\left(\overline{f\left(M^{N_1 N_2}\right)}_{\sigma_2}\right) \leq \varepsilon_2,$$

and so on.

In general, the condition on the growth of $N_1, \dots, N_k, \dots$ is

$$\text{g)} \quad N_k > \max\left\{\frac{J_{N_1 \dots N_{k-1}} \cdot 80}{\varepsilon_k}, \frac{1}{2 \cdot \sigma_{k-1}}\right\}.$$

Lemma 5. Let

$$M = \lim_{k \to \infty} M^{N_1 \dots N_k},$$

where the sequence $N_1, \dots, N_k, \dots$ satisfies condition g). Then $\operatorname{mes}(f(M)) = 0$ for any contracted mapping f onto a line.

Since $N_k \leq \dfrac{1}{2\sigma_{k-1}}$, it follows, by condition b), that

$$\delta_0^k \leq \frac{1}{2N_k} \leq \frac{1}{4}\sigma_{k-1}, \qquad \delta_0^{k+1} \leq \frac{1}{4}\sigma_k \leq \frac{1}{16}\sigma_{k-1};$$

then the maximal deviation of the set $M^{N_1 \ldots N_k}$ from $M^{N_1 \ldots N_{k-1}}$ is less than $\frac{1}{2}\sigma_{k-1}$, and that of the set $M^{N_1 \ldots N_{k+1}}$ from $M^{N_1 \ldots N_{k-1}}$ is less than $(\frac{1}{2}+\frac{1}{4})\sigma_{k-1}$, and, in general,

$$M \subset \left(M^{N_1 \ldots N_{k-1}}\right)_{\sigma_{k-1}}.$$

But

$$\operatorname{mes}\left(\overline{f\left(M^{N_1 \ldots N_{k-1}}\right)_{\sigma_{k-1}}}\right) \leq \varepsilon_{k-1}.$$

Hence, by Whitney' s theorem on the extension of a function with Lipschitz condition with constant 1 on the set M to the whole plane with Lipschitz constant C, it follows that for any contracted mapping onto a line

$$\operatorname{mes}(f(M)) \leq C\varepsilon_{k-1}, \qquad \text{i.e.} \quad \operatorname{mes}(f(M)) = 0.$$

It follows from Lemmas 3 and 5 that for the set M the length $l(M)$ and the minimal linear measure $L(M)$ are incommensurable.

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CITED LITERATURE

1. A. N. Kolmogoroff, *Math. Ann.*, **107**, 351 (1932).

Note: Figure translations are in progress. See original paper for figures.

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