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Abstract

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On the Length of a Cycle in an n-Dimensional Unit Cube

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1. A **cycle** in the n-dimensional unit cube* E^n is a subset S of the set E^n possessing the following properties: first, it is connected, i.e., for any two vertices α and β from S one can specify a number m and m vertices ${}^1\sigma, \dots, {}^m\sigma$ such that the vertices α and ${}^1\sigma$, ${}^1\sigma$ and ${}^2\sigma, \dots, {}^{m-1}\sigma$ and ${}^m\sigma$, ${}^m\sigma$ and α are adjacent; second, it is “one-dimensional”and “closed,”i.e., the neighborhood of an arbitrary vertex of the set S contains exactly two vertices of the set S. The **length** of a cycle is the number of vertices in the set S.

The problem of the maximal length of a cycle in the n-dimensional unit cube was formulated by Yu. I. Zhuravlev (^{1}) in connection with the study of algorithms for simplifying disjunctive normal forms (d.n.f.).** Yu. I. Zhuravlev observed that, in order to estimate the dependence between the “performance” of an algorithm A for simplifying d.n.f.’ s and the “volume” of that “part” of the reduced d.n.f. of a given function which is “surveyed”by the algorithm A “in one step,” it is essential to determine how the numerical function $C(n)$—the greatest length of a cycle in the n-dimensional unit cube E^n—behaves. Being related to the well-known “coding problem,”the question of constructing a cycle of greatest length is also of independent interest. By the definition of a cycle, $C(n) \leq 2^n$.

Theorem 1. *For every* $n = 2^m,\ m = 3, 4, \dots,$ *one can construct a cycle in* E^n *whose length is* $2^n/n$.

Theorem 2. *For arbitrary* $n = 3, 4, \dots,$ *one can construct a cycle in* E^n *whose length is* $(1 - \varepsilon(n))\, 2^{n-1}/n,\ \varepsilon(n) \to 0$ *as* $n \to \infty$.

Theorem 3. *Let* $M(n)$ *be the number of distinct cycles in* E^n *of length not less than* $2^n/n,\ n \geq 2^{-4}$. *Then* $M(n) \geq 2^{2^n(1-\varepsilon)},\ \varepsilon \to 0$ *as* $n \to \infty$.

Below we shall consider the main points in the proof of Theorem 1, carried out by a direct construction of a cycle of the indicated length. This construction rests on properties of splittings (see below) of certain d.n.f.’ s and arose in connection with the study of the question of the maximal possible difference between the complexity of minimal and dead-end d.n.f.’ s (^{3}). In § 2, in order to describe the construction from given cycles of a cycle of greater length, the notion of a pseudocycle is introduced. In § 3 the means by which the pseudocycle will be constructed are considered. In § 4 an outline is given of the proof of Theorem 1.

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* By definition, E^n is the set of all n-digit strings composed of zeros and ones. Each such string is called a **vertex** of the cube E^n. Two vertices of the cube E^n are called **adjacent** if they differ from one another in only one digit. The **neighborhood** of the vertex $\sigma = (\sigma_1, \dots, \sigma_n)$ of the cube E^n is the set of all vertices of the cube E^n adjacent to the vertex σ. An unordered pair of adjacent vertices of the cube E^n will be called an **edge** and denoted by p; the set of all edges of the cube E^n will be denoted by $\mathfrak{R}^p$. For a vertex $\sigma = (\sigma_1, \dots, \sigma_n)$ of the cube E^n we define $|\sigma| = \sigma_1 \oplus \cdots \oplus \sigma_n$, where $\oplus$ denotes addition mod 2.

** The basic concepts of the theory of d.n.f.' s are set forth in $(^2)$. We shall say that the vertex $\sigma = (\sigma_1, \dots, \sigma_n)$ is a vertex of the d.n.f. $\Phi(x^n)$ if $\Phi(\sigma_1, \dots, \sigma_n) \equiv 1$. The **distance** ρ between vertices σ and τ of the cube E^n is the number of positions in which these vertices do not coincide. The distance between d.n.f.' s $\Phi_1(x^n)$ and $\Phi_2(x^n)$ is the number

$$\min_{\sigma \in N_1,\ \tau \in N_2} \rho(\sigma, \tau),$$

where N_1, N_2 are, respectively, the sets of vertices of the d.n.f.' s Φ_1 and Φ_2.

2. **Definition.** A D.N.F. Φ is called **connected** if, for any two of its vertices α and β, one can indicate a number m and m conjunctions $K_1, \dots, K_m$ from Φ such that α and β belong respectively to the conjunctions K_1 and K_m, and $K_i K_{i+1} \neq 0$ for all $i = 1, \dots, m-1$.

The following definition of a cycle differs from the definition given above only in form.

Definition. A **cycle** is a connected D.N.F. $S(x^n)$, all conjunctions of which are one-dimensional*, and, moreover, the neighborhood of each vertex of the D.N.F. $S(x^n)$ contains exactly two vertices of the D.N.F. $S(x^n)$. The **length** of the cycle S is the number of conjunctions of the D.N.F. S; it will be denoted by $l(S)$. It is not difficult to verify that a cycle is a reduced D.N.F. and that the length of a cycle is always even.

Definition. A **path** between the vertices α and β of the cube E^n is a connected D.N.F. $P(x^n)$, all conjunctions of which are one-dimensional and to which the vertices α and β belong; moreover, the neighborhood of an arbitrary vertex v of the D.N.F. $P(x^n)$ contains only one vertex of the D.N.F. $P(x^n)$ when v is one of the vertices α or β, and contains exactly two vertices of the D.N.F. $P(x^n)$ when $v \neq \alpha,\ v \neq \beta$. The vertices α and β of the path $P(x^n)$ will be called **boundary** vertices, and the other vertices of the path $P(x^n)$–**internal** vertices. The **length** of the path P is the number of conjunctions of the D.N.F. P, and will be denoted by $l(P)$.

Definition. Let ${}^1S, \dots, {}^tS$ be nonintersecting cycles. Suppose that in the cycle ${}^iS$ two vertices ${}^i\alpha$ and ${}^i\beta$ are marked, which are not adjacent, and suppose that ${}^iP$ is a path between the vertex ${}^i\beta$ of the cycle ${}^iS$ and the vertex ${}^{i+1}\alpha$ of the cycle ${}^{i+1}S,\ i = 1, \dots, t$**. The D.N.F.

$$\bigvee_{i=1}^{t} {}^{i}S \vee \bigvee_{i=1}^{t} {}^{i}P$$

will be called a **pseudocycle**. The vertices ${}^{i}\alpha, {}^{i}\beta,\ i = 1, \dots, t$, will be called the **nodes** of the pseudocycle. A pseudocycle Π will be called **exact** if the neighborhood of any of its vertices v contains exactly three vertices of Π when v is one of the node vertices of Π, and contains exactly two vertices of Π when v is not a node vertex of Π.

Starting from a given exact pseudocycle, it is not difficult to construct a new cycle. To this end, observe that the vertices ${}^{i}\alpha$ and ${}^{i}\beta$ generate a decomposition of the cycle ${}^{i}S$ into two paths, which we shall denote by ${}^{i1}S$ and ${}^{i2}S$, and we agree that $l({}^{i1}S) \leq l({}^{i2}S)$. Namely, the paths ${}^{i1}S$ and ${}^{i2}S$ are composed of conjunctions of the cycle ${}^{i}S$, have boundary vertices ${}^{i}\alpha$ and ${}^{i}\beta$, and have no common internal vertices,

$$l({}^{i1}S) + l({}^{i2}S) = l({}^{i}S).$$

Proposition 1. If the pseudocycle

$$\Pi = \bigvee_{i=1}^{t} {}^{i}S \vee \bigvee_{i=1}^{t} {}^{i}P$$

is exact, then the D.N.F.

$$C = \bigvee_{i=1}^{t} {}^{i2}S \vee \bigvee_{i=1}^{t} {}^{i}P$$

is a cycle.

3. **Definition.** A **development** of the cube E^n is a connected D.N.F. of the function $1(x^n) \equiv 1$, all conjunctions of which are one-dimensional and each vertex of which belongs to exactly two conjunctions.

By induction on $n = 1, 2, \dots$ it is not difficult to verify that a development of the cube E^n exists. The number of conjunctions in a development of the cube is equal to 2^n. For example, the D.N.F.

$$x_1 \vee \bar{x}_1 \vee x_2 \vee \bar{x}_2$$

is a development of the cube E^2.

Recall that a set M of vertices of the cube E^p is called a **densely packed** (d.p.) $(p, 2l+1)$-code if the distance between any two arbitrary vertices from M is not less than $2l+1$, and if, for every vertex σ of the cube E^p, there is a vertex τ from M such that $\rho(\sigma,\tau) \leq l$. As is known (^{4}), d.p. $(p,3)$-codes exist for values $p = 2^q - 1$, $q = 2, 3, \dots$, and only for these values of p; the number of vertices in the p.u. $(p,3)$-code is equal to $2^p/(p+1)$.

Definition. We shall call the numbering of the vertices of a p.u. $(p,3)$-code **regular** if the distance between its i-th and $(i+1)$-st vertices is equal to 3, $i = 1, \dots, 2^p/(p+1)$.

Proposition 2. *For any $p = 2^q - 1$, $q = 2, 3, \dots$, one can construct a p.u. $(p,3)$-code having a regular numbering of the vertices.*

The proof follows, for example, from the inductive method of construction of p.u. $(p,3)$-codes proposed in (^{5}).

Definition. Let the variables u^s, z^p, w^p be pairwise distinct, $1 \leq s \leq p$, and let $\tau = (\tau_1, \dots, \tau_p)$ be an arbitrary vertex of E^p. By the **splitting of the conjunction $K = u_1^{\sigma_1} \cdots u_s^{\sigma_s}$ with respect to** τ we shall mean the conjunction

$$R_\tau[K] = z_1^{\sigma_1} w_1^{\sigma_1 \oplus \tau_1} \cdots z_s^{\sigma_s} w_s^{\sigma_s \oplus \tau_s}.$$

By the **splitting of a d.n.f. $\Phi(u^p)$ with respect to** τ we shall mean the d.n.f. $R_\tau[\Phi]$, made up of the splittings of the conjunctions of the d.n.f. $\Phi(u^p)$ with respect to τ.

4. Proof of Theorem 1. Starting from a partition of the set of vertices of the cube E^p and a p.u. $(p,3)$-code with regular numbering of the vertices, by means of splittings there are constructed in the cube E^{2p+2} "sufficiently many" "sufficiently long" cycles, and a method is given for joining them into a pseudocycle. Then, according to item 2, the pseudocycle is transformed into a cycle of the length indicated in Theorem 1. The construction is divided into six parts.

I. **Construction in the cube E^{2p} of $2^p/(p+1)$ nonintersecting cycles of length 2^{p+1}.** Denote by r_p the conjunction whose vertices are the edges p of the cube E^p, and only these vertices. Denote by $\mathfrak{R}^p$ such an arbitrary subset of the set $\mathfrak{R}^p$ of edges of the cube E^p that the d.n.f.

$$\bigvee_{p \in \widetilde{\mathfrak{R}}^p} r_p$$

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* That is, all variables $x_1, \dots, x_n$ enter into each conjunction except one of these variables.

** We agree that by the $(t+1)$-st element of an ordered set M of t elements we shall mean the first element o is a partition of the cube E^p; by $\tau = (\tau_1, \dots, \tau_p)$ denote an arbitrary vertex of the cube E^p. Consider the splitting of the d.n.f.

$$\bigvee_{p\in\widetilde{\mathfrak{R}}^p} r_p$$

–the d.n.f.

$$\Psi_\tau(z_p, w^p) = \bigvee_{p\in\widetilde{\mathfrak{R}}^p} R_\tau[r_p].$$

The d.n.f. Ψ_τ is connected, $I(\Psi_\tau) = 2^p$. A vertex σ of the d.n.f. Ψ_τ will be called a **breaking** vertex (respectively **plane**), if σ belongs to more than one (respectively only one) conjunction of the d.n.f. Ψ_τ. Each breaking vertex of the d.n.f. Ψ_τ belongs to exactly two conjunctions of the d.n.f. Ψ_τ. Of the four vertices of an arbitrary conjunction of the d.n.f. Ψ_τ, two vertices (not being adjacent) are breaking, the other two vertices are plane.

Pass from the d.n.f. Ψ_τ to a new d.n.f., which we denote by Φ_τ. Let $R_\tau[r_p]$ be an arbitrary conjunction of the d.n.f. Ψ_τ, and let $C_{p,\tau}$ be either one of the two paths of length 2 between the breaking vertices of the conjunction $R_\tau[r_p]$. The internal vertex of this path is a plane vertex of the conjunction $R_\tau[r_p]$. Put

$$\Phi_\tau(z^p, w^p) = \bigvee_{p\in\widetilde{\mathfrak{R}}^p} C_{p,\tau}.$$

Proposition 3. *The d.n.f.* Φ_τ *is a cycle,* $I(\Phi_\tau) = 2^{p+1}$. In what follows we shall assume that $p = 2^q - 1$, $q = 2, 3, \dots$.

Proposition 4. *Let* τ *and* ν, $\tau \neq \nu$, *be vertices of a p.u.* $(p, 3)$*-code. Then* $\Phi_\tau \cdot \Phi_\nu \equiv 0$.

Propositions 3 and 4 mean that in the cube E^{2p} one can construct $2^p/(p+1)$ nonintersecting cycles of length 2^{p+1}.

We pass to a refinement of the form of the initial partition and of the cycles. Let C^p be a p.u. $(p, 3)$-code with regular numbering of the vertices, t the number of vertices of the code C^p; $t = 2^p/(p+1)$, $i = 1, \dots, t$; i_τ the i-th vertex of the code C^p, with ${}^1\tau = (0, \dots, 0)$. By definition $|{}^i\tau| = |{}^{i+1}\tau| \oplus 1$. It is not difficult to verify that one can construct such a partition of the cube E^p, three conjunctions of which form a path ^{iQ} of length 3 between the vertices $(0, \dots, 0)$ and ${}^i\tau \oplus {}^{i+1}\tau$ of the cube E^p. Denote such a partition by ^{iR}. Denote

by ${}^i\mathfrak{R}$ such a subset of the set $\mathfrak{R}^p$ of edges of the cube E^p that $\bigvee_{\rho\in{}^i\mathfrak{R}} r_\rho$ is ^{iR}. Put ${}^{iS}(z^p, w^p) = \bigvee_{\rho\in{}^i\mathfrak{R}} C_{\rho,{}^i\tau}$. In other words, the d.n.f. ^{iS} is such a cycle Φ_τ, in the construction of which ${}^i\tau$ is taken as the vertex τ, and ^{iR} is taken as the initial unfolding; $l({}^{iS}) = 2p + 1$, $i = 1, \dots, t$.

Proposition 5. If $i \neq j$, $i, j = 1, \dots, t$, then $\rho({}^{iS}, {}^{jS}) \geq 1$, if $i \not\equiv j \pmod 2$; $\rho({}^{iS}, {}^{jS}) \geq 2$, if $i \equiv j \pmod 2$.

II. **Construction of nodal vertices.** Put

$$^{i}\alpha = (0, {}^{i}\tau) = (0, \dots, 0, {}^{i}\tau, \dots, {}^{i}\tau);$$

$$^{i}\beta = ({}^{i}\tau \oplus {}^{i+1}\tau, {}^{i+1}\tau) = ({}^{i}\tau_1 \oplus {}^{i+1}\tau_1, \dots, {}^{i}\tau_p \oplus {}^{i+1}\tau_p, {}^{i+1}\tau_1, \dots, {}^{i+1}\tau_p).$$

By construction the vertices ${}^{i}\alpha$ and ${}^{i}\beta$ belong to the cycle ^{iS}, $\rho({}^{i}\beta, {}^{i+1}\alpha) = 3$, $i = 1, \dots, t$. Further, the vertices ${}^{i}\alpha$ and ${}^{i}\beta$ generate a decomposition of the cycle ^{iS} into two paths, which we denote by ^{iS} and ${}^{i2}S$ (see the end of item 2). It turns out that $l({}^{i1}S) = 6$, $l({}^{i2}S) = 2p + 1 - 6$. This follows from the fact that, in constructing the cycle ^{iS}, we started not from an arbitrary unfolding of the cube E^p, but specifically from ^{iR}. In this case the path ^{iQ} of length 3 between $(0, \dots, 0)$ and ${}^{i}\tau \oplus {}^{i+1}\tau$ becomes the path ${}^{i1}S$ of length 6 between ${}^{i}\alpha$ and ${}^{i}\beta$.

III. **Construction of the paths ^{iP} between the vertices ${}^{i}\beta$ and ${}^{i+1}\alpha$ of the cycles ^{iS} and ${}^{i+1}S$.** Consider the 3-dimensional subcube of the cube E^{2p} containing the vertices ${}^{i}\beta$ and ${}^{i+1}\alpha$. Denote this subcube by ^{iK}. The vertices of any path joining ${}^{i}\beta$ and ${}^{i+1}\alpha$ and having length 3 belong to ^{iK}.

Proposition 6. Among the vertices of the cube ^{iK} adjacent to ${}^{i+1}\alpha$, there is one which does not belong to any of the cycles ^{jS}, $j = 1, \dots, t$ (we denote this vertex by ${}^{i}\xi$).

As the path ^{iP} we take an arbitrary path of length 3 between the vertices ${}^{i}\beta$ and ${}^{i+1}\alpha$, one of whose internal vertices is ${}^{i}\xi$. It is not difficult to verify that, for $i \neq j$, $\rho(^{iP}, {}^{jP}) \geq 3$.

IV. The d.n.f.

$$\Pi(z^p, w^p) = \bigvee_{i=1}^{t} {}^{iS} \vee \bigvee_{i=1}^{t} {}^{iP}$$

is a pseudocycle with nodal vertices ${}^{i}\alpha$ and ${}^{i}\beta$, $i = 1, \dots, t$. The pseudocycle Π is not exact.

V. **Transformation of the pseudocycle $\Pi(z^p, w^p)$.** Put

$$\Pi^*(z^p, w^p, x, y) = \bigvee_{i=1}^{t} {}^{i}S^* \vee \bigvee_{i=1}^{t} {}^{i}P^*,$$

where ${}^{i}S^*$ is the d.n.f. of the formula

$${}^{i}Sxy^{|{}^{i}\tau|}, \quad {}^{i}P^*$$

is the d.n.f. of the formula

$${}^{i}By^{|{}^{i}\tau|} \vee {}^{iBx} \vee {}^{i}Pxy^{|{}^{i+1}\tau|} \vee {}^{i+1}Ay^{|{}^{i+1}\tau|},$$

$${}^{iB} = \bigwedge_{j=1}^{p} z_j^{{}^{i}\beta_j} w_j^{{}^{i}\beta_{p+j}}, \quad {}^{i+1}A = \bigwedge_{j=1}^{p} z_j^{{}^{i+1}\alpha_j} w_j^{{}^{i+1}\alpha_{p+j}}.$$

The d.n.f. Π^* is an **exact** pseudocycle with nodal vertices

$$^{i}\alpha^* = (^{i}\alpha_1, \dots, {}^{i}\alpha_{2p}, 1, |^{i}\tau|)$$

and

$$^{i}\beta^* = (^{i}\beta_1, \dots, {}^{i}\beta_{2p}, 1, |^{i}\tau|), \quad i = 1, \dots, t.$$

The proof of the exactness of Π^* rests on Proposition 5 (item I) and on the properties of the paths ^{iP} (item III).

VI. **Passage from the pseudocycle** Π^* **to a cycle.** By virtue of item II of the present proof, the vertices $^{i}\alpha^*$ and $^{i}\beta^*$ generate a decomposition of the cycle $^{i}S^*$ into two paths $^{i1}S^*$ and $^{i2}S^*$, $l(^{i1}S^*) = 6$, $l(^{i2}S^*) = 2p+1-6$, $i = 1, \dots, t$. According to Proposition 1, the d.n.f.

$$C(z^p, w^p, x, y) = \bigvee_{i=1}^{t} {}^{i2}S^* \vee \bigvee_{i=1}^{t} {}^{i}P^*$$

is a cycle,

$$l(C) = t(l(^{i2}S^*) + l(^{i}P^*)) = t(2p + 1 - 6 + 6) = 2^{2p+1}/(p + 1) = 2^n/n,$$

where $n = 2p + 2 = 2^{q+1}$, $q = 2, 3, \dots$. Theorem 1 is proved.

We omit the consideration of Theorems 2 and 3.

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CITED LITERATURE

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