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Abstract

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MATHEMATICS

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ON THE BENDING OF A SURFACE WITH BOUNDARY

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It is known that a surface of positive curvature with boundary admits continuous bendings. In this case the boundary of the surface is deformed. The deformation of the boundary, however, is subject to a number of conditions. E. P. Sen' kin established ([2]) that under a bending of a surface with piecewise-smooth boundary, two pairs of points will be found on the boundary for which the spatial distances between them respectively increase and decrease. In our paper ([4]) we showed that under a bending of a surface with smooth boundary, two points will be found on the boundary at which the curvature of the boundary respectively decreases and increases. For surfaces with piecewise-smooth boundary this result, generally speaking, is not true. In the present paper we study the behavior of various characteristics of the boundary under bendings of surfaces. In particular, it is established that under a bending of a surface, two pairs of points A_1, A_2 and B_1, B_2, arranged in the order A_1, B_1, A_2, B_2 in a single traversal of the contour, will be found on the boundary, such that the curvature of the boundary at them respectively increases and decreases under the bending. First, a nonlinear boundary-value problem is considered for the fundamental equations of the theory of surfaces. The proof of the formulated theorems is carried out by the methods set forth in the book ([1]).

§ 1. We consider simply connected surfaces of strictly positive Gaussian curvature up to the boundary. We assume that the surfaces belong to the class $D_{3,p}$, $p > 2$ (the radius vector of the surface $\mathbf{r}(u, v)$ has three generalized derivatives in the sense of Sobolev, summable with exponent p). The boundary of the surface is assumed to be a simple smooth closed curve of class C^1_μ, $0 < \mu < 1$. Bendings of the surface S are considered in the class $D_{3,p}$, $p > 2$.

§ 2. Let a surface S with boundary $\mathcal{L}$ be given. Introduce on it an isothermally conjugate parametrization u, v. Let the surface S be bent into a surface S^*. The Gauss and Codazzi equations for the surface S^* can be written in the form ([3])

$$\partial_{\bar z} w + A_1(z, w, \bar w_z) w + B_1(z, w, \bar w_z) \bar w = 0, \qquad z = u + iv,$$

where $w = w(z)$ is the unknown function which, following the terminology of I. N. Vekua ([1]), we shall call the **complex function of the bending**; A_1 and B_1 are known functions of their arguments, belonging to the class L_p, $p > 2$, if $w \in D_{1,p}$, $p > 2$.

Statement of Problem A. Let D be the unit disk with boundary Γ. Find in the domain D a complex function of the bending $w(z)$, continuously extendable to the contour Γ, belonging to the class $D_{1,p}$, $p > 2$, and satisfying the boundary condition

$$\mathrm{Re}\{\overline{\lambda(t)} w(t)\} + \Phi(w; t) = 0, \qquad t \in \Gamma,$$

where the function $\lambda(t)$ is given and belongs to the class $C_\sigma(\Gamma)$, $\sigma \geq (p-2)/p$, $p > 2$; $|\lambda(t)| = 1$. Concerning the nonlinear part $\Phi(w; t)$ we make the following assumptions. The real-valued function $\Phi(w; t)$, for fixed $w(t)$ of class $C_\alpha(\Gamma)$, $0 < \alpha < 1$, as a function of t, belongs to the class $C_\alpha(\Gamma)$, $0 < \alpha < 1$, and with respect to $w = U + iV$ satisfies the condition:

$$\Phi(w; t) = \Phi_0(w; t) + \Phi_1(w; t) U + \Phi_2(w; t) V,$$

where the real functions $\Phi_i(w; t)$, $i = 0, 1, 2$, as functions of two variables, belong to the class C_α, $0 < \alpha < 1$. We require that the functions Φ_1 and Φ_2 satisfy the conditions:

$$\Phi_1(0; t) \equiv 0; \qquad \Phi_2(0; t) \equiv 0; \qquad \Phi_1^2(w; t) + \Phi_2^2(w; t) < 1, \qquad t \in \Gamma,$$

uniformly in w, and that the function $\Phi_0(w; t)$, for each fixed function $w(t)$ of class $C_\alpha(\Gamma)$, $0 < \alpha < 1$, have no more than two changes of sign on the contour Γ, i.e., for each function $w(t)$ there exist points t_1 and t_2 that divide the contour Γ into two parts Γ_1 and Γ_2, $\Gamma = \Gamma_1 + \Gamma_2$, such that $\Phi_0(w; t) \geqslant 0$ for $t \in \Gamma_1$; $\Phi_0(w; t) \leqslant 0$ for $t \in \Gamma_2$.

We shall call the number

$$n = \frac{1}{2\pi} \Delta_\Gamma \arg \lambda(t)$$

the **index of the problem**.

Theorem 1. *For $n < 0$, problem A in the class $D_{1,p}(D)$, $p > 2$, has no nonzero solutions.*

Proof. We shall show that if a solution of problem A exists, then it is identically equal to zero. Indeed, if a solution $w(z)$ exists and belongs to the class $D_{1,p}$, $p > 2$, then, according to (3), it can be represented in the form

$$w(z) = \varphi(z)e^{\omega(z)},$$

where $\varphi(z)$ is holomorphic in D, $\omega \in C_\alpha(E)$, $\alpha = (p-2)/p$, $p > 2$.

The function $\varphi(z)$ is a solution of the boundary-value problem

$$\mathrm{Re}\{\overline{\lambda_1(t)}\varphi(t)\} + \Phi_0(\varphi e^{\omega}; t) = 0,$$

where $\mathrm{Ind}\,\lambda_1(t) = n < 0$, and the function Φ_0, by assumption, changes sign twice on the contour Γ. Make a conformal mapping of the domain D onto itself in such a way that the point t_1 goes into the point $\xi_1 = 1$, and the point t_2 into the point $\xi_2 = -1$. Let $\zeta = \zeta(z)$ effect this mapping. Then $\xi_1 = \zeta(t_1)$, $\xi_2 = \zeta(t_2)$, $\varphi(z) = \varphi[z(\zeta)] \equiv \varphi_1(\zeta)$;

$$\Phi_0(\varphi(t)e^{\omega(t)}; t) = \Phi_0(\varphi[t(\xi)]e^{\omega[t(\xi)]}; t(\xi)) = \Phi_0(\varphi_1(\xi)e^{\omega_1(\xi)}; \xi), \quad \xi \in \Gamma_1,$$

where Γ_1 is the image of the contour Γ in the ζ-plane. In this case

$$\Phi_0(\varphi_1(\xi)e^{\omega_1(\xi)}; \xi) \geqslant 0 \quad \text{for } 0 \leqslant \arg\xi \leqslant \pi,$$

$$\Phi_0(\varphi_1(\xi)e^{\omega_1(\xi)}; \xi) \leqslant 0 \quad \text{for } \pi \leqslant \arg\xi \leqslant 2\pi.$$

In the ζ-plane we obtain the boundary-value problem:

$$\mathrm{Re}\{\overline{\lambda_2(\xi)}\varphi_1(\xi)\} + \Phi_0(\varphi_1 e^{\omega_1}; \xi) = 0; \qquad \xi \in \Gamma_1,$$

with $\mathrm{Ind}\,\lambda_2(\xi) = n < 0$. The solution of the problem satisfies a system of $2|n|+1$ integral equations

$$\varphi_1(\zeta) = \zeta^{-n} e^{i\gamma(\zeta)} \frac{1}{2\pi} \int_0^{2\pi} e^{\omega_1'(\sigma)} \frac{\Phi_0(\varphi_1 e^{\omega_1}; \sigma)}{|\lambda_2(\sigma)|} \frac{e^{i\sigma} + \zeta}{e^{i\sigma} - \zeta}\, d\sigma;$$

$$\int_0^{2\pi} \frac{e^{\omega_1'(\sigma)}}{|\lambda_2(\sigma)|} \Phi_0(\varphi_1 e^{\omega}; \sigma) e^{ik\sigma}\, d\sigma = 0, \qquad k = 0, 1, \ldots, -n-1,$$

where

$$\gamma(\zeta) = \omega_2' + i\omega_1' = \frac{1}{2\pi}\int_0^{2\pi}[\arg\lambda_2 - n\sigma]\frac{e^{i\sigma}+\zeta}{e^{i\sigma}-\zeta}\,d\sigma; \qquad \sigma = \arg\xi; \quad \xi \in \Gamma_1.$$

For $k = 1$ in this system we have the equation

$$\int_0^{2\pi} \frac{e^{\omega_1'(\sigma)}}{|\lambda_2(\sigma)|}\Phi_0(\varphi_1 e^{\omega};\sigma)\sin\sigma\,d\sigma = 0.$$

Since $\Phi_0(\varphi_1 e^{\omega_1};\sigma)\sin\sigma \geqslant 0$ for $0 \leqslant \sigma \leqslant 2\pi$, the last equality is possible only when $\Phi_0(\varphi_1 e^{\omega_1};\sigma) \equiv 0$, but then from the first equation of the system it follows that $\varphi(z) \equiv 0$, whence we obtain $w(z) \equiv 0$.

3. Let us prescribe on the contour $\mathcal{L}$ two functions $\lambda(s)$, $\mu(s)$ of class C_α, $0 < \alpha < 1$, $\lambda^2(s) + \mu^2(s) \neq 0$, and a direction field R having no singular points. Denote by k_{n_R} the normal curvature and by τ_{g_R} the geodesic torsion of the po-

surface along the edge in the direction R. Let the surface S be isometrically transformed into a surface S^*. Then the quantities k_{n_R} and τ_{g_R} receive, respectively, certain increments Δk_{n_R} and $\Delta\tau_{g_R}$. Consider the expression

$$\sigma(s) = \lambda(s)\Delta k_{n_R} + \mu(s)\Delta\tau_{g_R}. \tag{*}$$

To each nontrivial isometric transformation of the surface S, formula $(*)$ assigns a certain function $\sigma(s)$, belonging to the class C_α, $0 < \alpha < 1$. In passing from one isometric transformation to another, the function $\sigma(s)$, generally speaking, changes. We consider conditions for the existence of nontrivial isometric transformations of the surface S satisfying condition $(*)$, where $\sigma(s)$ is a prescribed function of class C_α, $0 < \alpha < 1$.

In [4] the deficiency $v_R(S)$ of the surface with respect to the field R is defined. In what follows it is convenient for us to use the notion of the index of a surface with respect to the field R, defined by the formula $j_R(S) = v_R(S) + 2$. It can be shown that $j_R(S)$ is a topological invariant. It is computed directly for the field R given on the surface, and does not depend on isometric transformations of the latter.

With the aid of Theorem 1, the following theorem is proved.

Theorem 2. *Let* $\mathrm{Ind}(\mu;\lambda) < j_R(S)$*, and let the function* $\sigma(s)$ *prescribed on the contour* $\mathcal{L}$ *have no more than two changes of sign. Then there exist no nontrivial isometric transformations of the surface* S *satisfying condition* $(*)$.

A number of corollaries follow from Theorem 2.

Corollary 1. *Under a bending of a surface of positive curvature with smooth edge $\mathcal{L}$, the increment of the curvature of the edge along $\mathcal{L}$ has at least four changes of sign.*

Indeed, in condition $(*)$ choose $\lambda = 1$, $\mu = 0$, and take the field R to coincide with the field of tangents to the curve $\mathcal{L}$. Then $\operatorname{Ind}(\mu; \lambda) = 0$, $j_R(S) = 2$, and consequently, by Theorem 2, the increment of the normal curvature Δk_n of the strip of the edge has at least four changes of sign. Since

$$2k\Delta k + \Delta k^2 = 2k_n \Delta k_n + \Delta k_n^2,$$

where k and Δk are, respectively, the curvature of the edge and its increment under bending, $k_n \neq 0$, $k_n + \Delta k_n \neq 0$, the assertion is proved.

Corollary 2. *Under a bending of a surface of positive curvature with smooth edge $\mathcal{L}$, the increment of the geodesic torsion of the strip of the edge along $\mathcal{L}$ has at least four changes of sign.*

Theorem 3. *Let* $\operatorname{Ind}(\mu; \lambda) \geq j_R(S)$ *and* $C_\alpha(\sigma; \mathcal{L}) < \varepsilon$, *where* ε *is a number determined by the surface* S. *Then for the given surface* S *there exists a* $2\operatorname{Ind}(\mu; \lambda) - 2j_R(S) + 1$*-parameter family* S_σ *of isometric surfaces satisfying condition* $(*)$. *All surfaces of the family admit continuous bendings into one another with preservation of condition* $(*)$. *In this case the initial surface* S *may or may not be included in the family* S_σ. *The surface* S, *however, admits continuous bendings into any surface of the family* S_σ.

Let l_ε be the set of points of the edge of positive linear measure, $\operatorname{mes} l_\varepsilon = \varepsilon > 0$, where ε is an arbitrarily small number. Let, further, $\mathcal{L}_\varepsilon = \mathcal{L} - l_\varepsilon$.

From Theorem 3 follows the following assertion:

Corollary 3. *A surface of positive curvature admits continuous bendings that preserve along* $\mathcal{L}_\varepsilon$ *the curvature of the edge.*

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Note: Figure translations are in progress. See original paper for figures.