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Abstract

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ON THE DIVERGENCE OF FOURIER SERIES OF CONTINUOUS FUNCTIONS WITH RESPECT TO A REARRANGED TRIGONOMETRIC SYSTEM

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Let $\{\varphi_n(x)\}$ be an arbitrary complete orthonormal system of functions defined on the interval $[0, 1]$. In the work of A. M. Olevskii ([2]) the following result was obtained:

There exists a continuous function $f(x)$ whose Fourier series

$$\sum_{n=1}^{\infty} c_n \varphi_n(x)$$

after a certain rearrangement of its terms diverges almost everywhere on $[0, 1]$.

Considering the proof of this proposition, we see that in fact unbounded divergence almost everywhere of the rearranged Fourier series has been proved.

In the case of the trigonometric system the result of A. M. Olevskii can be strengthened; namely, it will be proved here that there exists a pair of conjugate continuous functions and such a rearrangement of the trigonometric system that the Fourier series of these functions with respect to the rearranged trigonometric system will diverge unboundedly almost everywhere.

Lemma 1. Let two functional series be given,

$$\sum_{n=0}^{\infty} f_n(x), \qquad \sum_{n=0}^{\infty} \varphi_n(x),$$

where all the functions $f_n(x)$, $\varphi_n(x)$, $n = 0, 1, ...$, are continuous on the finite interval $[a, b]$. Then, if the first series diverges unboundedly on the set E_1, and the second series diverges unboundedly on the set E_2, then for every sequence of closed sets

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots \subseteq (E_1 \cup E_2)$$

there exists an increasing sequence of indices $\{R_k\}$, $k = 1, 2, \ldots$, such that the series

$$\sum_{k=1}^{\infty} \left\{ \sum_{n=R_k+1}^{R_{k+1}} f_n(x) + \sum_{n=R_k+1}^{R_{k+1}} \varphi_n(x) \right\}$$

will have the following property: for each index k and every $x \in F_k$ there is an index $i_k(x)$ such that $R_k + 1 \le i_k(x) \le R_{k+1}$, and one of the two inequalities holds:

$$\left| \sum_{n=R_k+1}^{i_k(x)} f_n(x) \right| > k, \qquad \left| \sum_{n=R_k+1}^{i_k(x)} \varphi_n(x) \right| > k.$$

Proof. By the hypothesis of the lemma, for each point $x_1 \in F_1 \cdot E_1$ there is an index $p_1(x_1)$ such that

$$\left| \sum_{n=0}^{p_1(x_1)} f_n(x_1) \right| > 1. \tag{1}$$

Similarly, for any point $x_2 \in F_1 \cdot E_2$ there is a number $q_1(x_2)$ such that

$$\left| \sum_{n=0}^{q_1(x_2)} \varphi_n(x_2) \right| > 1. \tag{2}$$

Thus, for every point $x \in F_1$ either (1) or (2) holds. Since the functions $f_n(x)$ and $\varphi_n(x)$ are continuous on the interval $[a, b]$, for every point $x \in F_1$ there is a neighborhood $\delta(x)$ such that inequality (1) or (2) will hold for all points $\xi \in \delta(x)$. Consequently, the closed set is covered by the infinite system of neighborhoods $\{\delta(x)\}$.

Choose from the infinite cover $\{\delta(x)\}$ a finite subcover $\{\delta(x_s)\}$, $s = 1, 2, \ldots, m$. We shall assume that the points $x_1, x_2, \ldots, x_r$, and only they, from the system $\{x_s\}$, $s = 1, 2, \ldots, m$, belong to the set $F_1 \cdot E_1$. Then the remaining points $x_{r+1}, x_{r+2}, \ldots, x_m$ belong to the set $E_2 \cdot F_1$.

Put

$$R_1 = \max\{p_1(x_1), \ldots, p_1(x_r), q_1(x_{r+1}), \ldots, q_1(x_m)\}.$$

Then for any point $x \in F_1$ there is a number $i_1(x)$ such that, if $x \in \bigcup_{s=1}^{r} \delta(x_s)$, then

$$\left|\sum_{n=0}^{i_1(x)} f_n(x)\right| > 1, \qquad 0 \le i_1(x) \le R_1,$$

while if $x \in \overline{\bigcup_{s=1}^{r} \delta(x_s)}$, then

$$\left|\sum_{n=0}^{i_1(x)} \varphi_n(x)\right| > 1, \qquad 0 \le i_1(x) \le R_1.$$

The series

$$\sum_{n=-R_1+1}^{\infty} f_n(x), \qquad \sum_{n=R_1+1}^{\infty} \varphi_n(x)$$

satisfy the conditions of Lemma 1; therefore, repeating the preceding arguments with the obvious changes and using induction, we complete the proof of Lemma 1.

Let $n_0 = 0$, and let $n_k = n(k)$ be a one-to-one mapping of the natural-number sequence onto itself. By partial sums of the series in the rearranged trigonometric system

$$\sigma(x) = \sum_{k=0}^{\infty} \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x\right)$$

we shall mean the expressions

$$S_m(\sigma, x) = \sum_{k=1}^{m} \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x\right), \qquad m = 0, 1, \ldots .$$

Two rearranged trigonometric systems $\{\cos n_k x, \sin n_k x\}_{k=0}^{\infty}$ and $\{\cos m_k x, \sin m_k x\}_{k=0}^{\infty}$ will be regarded as distinct if $n_k \ne m_k$ for at least one natural number k.

Lemma 2. Let M be a natural number. Then for any polynomial

$$T(x) = \sum_{k=0}^{N} \left(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x\right)$$

in a rearranged trigonometric system there is a polynomial

$$t(x) = \sum_{k=Q}^{P} \left(c_{m_k} \cos m_k x + d_{m_k} \sin m_k x\right)$$

with respect to some other rearranged trigonometric system possessing the following properties:

1)

$$\min_{Q \le k \le P} m_k > M;$$

2) for every point $x \in [0, 2\pi)$,

$$|t(x)| \le |T(x)|, \qquad |\tilde{t}(x)| \le |T(x)|;$$

3) for every point $x \in [0, 2\pi)$,

$$\frac{1}{4} \sin \frac{\pi}{8} \sup_k |S_k(T, x)| \le \sup_n |S_n(\tilde{t}, x)|.$$

Lemma 2 is a slight modification of one result of the author ([3]) and is proved analogously. For sufficiently large natural K, the required polynomial will be

$$t(x) = \frac{1}{2}(\cos Kx + \sin 3Kx)\, T(x).$$

Theorem. There exists a pair of conjugate continuous functions and such a rearrangement of the trigonometric system that the Fourier series of these functions with respect to the rearranged trigonometric system will diverge unboundedly almost everywhere.

Proof. By Olevskii' s theorem there exists a continuous function $f(x)$ and such a rearrangement of the trigonometric system that the Fourier series of this function with respect to the rearranged trigonometric system

$$\sum_{k=1}^{\infty} \left(a_{n_k} \cos n_{kx} + b_{n_k} \sin n_{kx} \right) \tag{3}$$

will diverge unboundedly almost everywhere.

By D. E. Menshov' s theorem ([1]), we decompose the function $f(x)$ into the sum of two continuous functions $f(x) = f_1(x) + f_2(x)$, such that their Fourier series

$$f_1(x) \sim \frac{a_0'}{2} + \sum_{k=1}^{\infty} \left(a_k' \cos kx + b_k' \sin kx \right), \tag{4}$$

$$f_2(x) \sim \frac{a_0''}{2} + \sum_{k=1}^{\infty} \left(a_k'' \cos kx + b_k'' \sin kx \right) \tag{5}$$

will contain subsequences of partial sums that converge uniformly respectively to $f_1(x)$ and $f_2(x)$. Let these subsequences be $S_{p_m}(f_1, x)$ and $S_{q_m}(f_2, x)$ ($m =$

$1,2,\ldots)$. Without changing notation, we shall assume that the subsequences of indices $\{p_m\}$ and $\{q_m\}$ are such that, for all $\mu \geq m$,

$$\max_{x\in[0,2\pi)} |S_{p_\mu}(f_1,x) - f_1(x)| < \frac{1}{2^m}, \qquad \max_{x\in[0,2\pi)} |S_{q_\mu}(f_2,x) - f_2(x)| < \frac{1}{2^m}. \quad (6)$$

Thus, the functions $f_1(x)$ and $f_2(x)$ are representable by uniformly convergent series:

$$f_1(x) = S_{p_1}(f_1,x) + \sum_{k=1}^{\infty}\{S_{p_{k+1}}(f_1,x) - S_{p_k}(f_1,x)\},$$

$$f_2(x) = S_{q_1}(f_2,x) + \sum_{k=1}^{\infty}\{S_{q_{k+1}}(f_2,x) - S_{q_k}(f_2,x)\}.$$

Rearranging the terms of the series (4) and (5), just as in (3), we obtain the series

$$\sum_{k=1}^{\infty}\left(a'_{n_k}\cos n_{kx} + b'_{n_k}\sin n_{kx}\right), \qquad \sum_{k=1}^{\infty}\left(a''_{n_k}\cos n_{kx} + b''_{n_k}\sin n_{kx}\right), \quad (7)$$

which together yield the series (3).

Let the set $E \subset [0,2\pi)$ be the set of points of unbounded divergence of the series (3). Choose a sequence of closed sets such that

$$F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots \subset E, \qquad mF_n \geq 2\pi - \frac{1}{n}.$$

On the basis of Lemma 1 we can form the series

$$\sum_{k=1}^{\infty}\left\{\sum_{m=R_k+1}^{R_{k+1}}\left(a'_{n_m}\cos n_m x + b'_{n_m}\sin n_m x\right) + \sum_{m=R_k+1}^{R_{k+1}}\left(a''_{n_m}\cos n_m x + b''_{n_m}\sin n_m x\right)\right\}$$

such that for any $x \in F_k$ either

$$|S'(x)| = \left|\sum_{m=R_k+1}^{i_k(x)}\left(a'_{n_m}\cos n_m x + b'_{n_m}\sin n_m x\right)\right| > k, \quad (8)$$

or

$$|S''(x)| + \left|\sum_{m=R_k+1}^{i_k(x)}\left(a''_{n_m}\cos n_m x + b''_{n_m}\sin n_m x\right)\right| > k.$$

Up to now the sequences $\{p_m\}$, $\{q_m\}$, and $\{R_m\}$ have been chosen independently of one another. We now choose from them subsequences $\{p_{m_i}\}$, $\{q_{m_i}\}$, and $\{R_{j_i}\}$ so that the following conditions are satisfied:

$$p_{m_i} + 1 < \min_{R_{j_i}+1 \le m \le R_{j_{i+1}}} n_m \le \max_{R_{j_i}+1 \le m \le R_{j_{i+1}}} n_m \le p_{m_{i+1}}, \tag{9}$$

$$q_{m_i} + 1 < \min_{R_{j_i}+1 \le m \le R_{j_{i+1}}} n_m \le \max_{R_{j_i}+1 \le m \le R_{j_{i+1}}} n_m \le q_{m_{i+1}}.$$

Choose two sequences of natural numbers $\{N_i\}$ and $\{M_i\}$ increasing so rapidly that the series

$$\sum_{i=1}^{\infty} \Big\{ (\cos N_i x + \sin 3N_i x) \left[S_{p_{m_{i+1}}}(f_1, x) - S_{p_{m_i}}(f_1, x) \right]$$

$$+ (\cos M_i x + \sin 3M_i x) \left[S_{q_{m_{i+1}}}(f_2, x) - S_{q_{m_i}}(f_2, x) \right] \Big\} \tag{10}$$

has no similar terms. By (6), the series (10) will be the Fourier series of a continuous function. By (6) and Lemma 2, the conjugate series will also be the Fourier series of a continuous function. Rearrange the terms of the series (10) in the square brackets so that the new order in which they follow is the same as in the series (7). Then for each $x \in F_{j_i}$ the rearranged series will contain the sums (see (9))

$$(\cos N_i x + \sin 3N_i x) S'(x), \qquad (\cos M_i + \sin 3M_i x) S''(x),$$

and by (8) will diverge unboundedly almost everywhere. The rearranged conjugate series will contain the sums

$$(\sin N_i x - \cos 3N_i x) S'(x), \qquad (\sin M_i x - \cos 3M_i x) S''(x),$$

and therefore will also diverge unboundedly almost everywhere.

The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.