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Abstract

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THEORY OF ELASTICITY

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ELASTIC-RIGID-PLASTIC TORSION OF A CYLINDRICAL ROD OF OVAL CROSS-SECTION

(Presented by Academician Yu. N. Rabotnov, 25 X 1962)

1. Let the cross-section F of the rod be bounded by a strictly convex contour Γ, having a tangent at each point, and let $Oxyz$ be a right rectangular Cartesian coordinate system such that the point $O \in F$, the axis Oz is directed parallel to the generator of the cylindrical surface, and the tangent to Γ at the point of intersection of Γ with the axis Ox is perpendicular to the axis Ox. We denote by: $\alpha > 0$ the relative angle of twist, occurring counterclockwise when viewed from the side of the positive direction of the axis Oz; G the shear modulus; k the plasticity constant, $k(2G\alpha)^{-1} = a$; $\sqrt{\Omega\pi^{-1}} = b$, where Ω is the area of $F + \Gamma$. Suppose that there is an elastic core–an area D with boundary L, lying entirely inside Γ. Denote by B the region between Γ and L; in this region the material is in a purely plastic state. The problem of elastic-rigid-plastic torsion (Problem I) is posed as follows ([1,2]):

Problem I. Find a simply connected region D with boundary L and a function* $\psi(x, y)$, defined and continuous in $F + \Gamma$ and having discontinuous partial derivatives ψ_x and ψ_y in $F + \Gamma$, if:

a) in D**

$$\psi_{xx} + \psi_{yy} = -2G\alpha, \qquad \psi_x^2 + \psi_y^2 < k^2;$$

b) in $L + B + \Gamma$

$$\psi_x^2 + \psi_y^2 = k^2; \tag{1.1}$$

c) on Γ

$$\psi|_\Gamma = 0. \tag{1.2}$$

2. Let R be an arbitrary point of Γ, and let $x_1 R y_1$ be the moving coordinate system formed by the tangent, directed in the direction of the positive traversal of Γ, and the inward normal to Γ at the point R. Let β be the

Fig. 1

Figure 1: Fig. 1

angle between Rx_1 and Ox, $\pi/2 \leq \beta \leq \pi/2$. The equation of the contour Γ can be represented in the form (([3]), p. 184):

$$x_\Gamma(\beta) = \frac{dM(\beta)}{d\beta}\cos\beta + M(\beta)\sin\beta, \qquad y_\Gamma(\beta) = \frac{dM(\beta)}{d\beta}\sin\beta - M(\beta)\cos\beta, \tag{2.1}$$

where $M(\beta) > 0$ is the support function of the contour Γ.

Definition. We shall say that a curve L, lying inside Γ, has property E if:

* In fact it is necessary to find $\psi_x \equiv \psi_x(x,y) = \partial\psi(x,y)/\partial x = -\tau_{yz}$, $\psi_y = \tau_{xz}$ (τ_{xz}, τ_{yz} are shear stresses); consequently, $\psi(x,y)$ is sought up to an additive constant.

** $\psi_{xx} = \psi_{xx}(x,y) = \partial^2\psi(x,y)/\partial x^2$, etc.

1) L can be represented by the equation

$$x_L(\beta) = x_\Gamma(\beta) - N(\beta)\sin\beta, \qquad y_L(\beta) = y_\Gamma(\beta) + N(\beta)\cos\beta, \tag{2,2}$$

where $N(\beta) > 0$ is a single-valued, periodic function with period 2π, continuous in β on $[\pi/2, 5\pi/2]$;

2) as β increases, L is traversed in the positive direction;

3) for any two distinct values of the angle β, $\beta_1, \beta_2 \in [\pi/2, 5\pi/2]$, the line segments R_1Q_1 and R_2Q_2, where $R_i \in \Gamma$ corresponds to β_i by (2,1), and $Q_i \in L$ by (2,2), $i = 1, 2$, have no common points*.

Theorem 1. There exists no more than one solution of Problem I for which L has property E.

Suppose that a solution of Problem I for which L has property E exists. We shall establish certain properties (1°—5°) of this solution, from which Theorem 1 will follow.

Fig. 1

1°. In $\Gamma + B + L$ the following relations hold, following from the solution by Cauchy' s method of the Cauchy problem for equation (1,1) under condition (1,2), taking (2,1) into account**:

$$\psi_x = -k\sin\beta, \qquad \psi_y = k\cos\beta; \tag{2,3}$$

$$-\psi_x y + \psi_y x = k\frac{dM(\beta)}{d\beta}; \tag{2,4}$$

$$\psi - x\psi_x - y\psi_y = kM(\beta). \tag{2,5}$$

2°. Everywhere in D,

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 \neq 0.$$

Proof. Let at the point $(x_0, y_0) \in D$

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 = 0.$$

Assume further that $\psi_{xy}(x_0, y_0) = \psi_{xx}(x_0, y_0) = 0$. Then (([5], pp. 428–430)) the harmonic function $\psi_x(x, y)$ takes on L the value $\psi_x(x_0, y_0)$ at least at four distinct points, which contradicts (2,3).

3°. The formulas $\xi = -\psi_x(x, y)/k$, $\eta = -\psi_y(x, y)/k$ realize a homeomorphic mapping of $D + L$ onto the disk $K + C:\ \xi^2 + \eta^2 = 1$ of the plane (ξ, η). The validity of this assertion follows from known theorems (([6], p. 26; ([7], p. 586)).

4°. Everywhere in D,

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 > 0.$$

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* Obviously, L is a simple chord-arc curve such that to each value of β there corresponds by (2,2) one and only one point $Q \in L$; $N(\beta)$ is the length of this segment in the coordinate system $x_1 R y_1$, where R is the point Γ corresponding to the given β.

** Taking into account the results of work ([4]) and using the equalities (2,3), (2,4), one can show that Problem I, for which L has property E, can be reduced to the following integral equation for the function $\gamma(t)$, $0 \le t \le 2\pi$:

$$\frac{k}{4\pi G a}\int_0^{2\pi} \cos[\gamma(t) - \gamma(\tau)]\,\mathrm{ctg}\,\frac{t-\tau}{2}\,d\tau = f\left(\gamma(t) + \frac{\pi}{2}\right),$$

where $f(\beta) = k\,dM(\beta)/d\beta$, $\gamma(t) = \beta - \pi/2$.

Proof. Suppose the contrary. Then the continuous vector field $\mathbf{v} = (\psi_x, \psi_y)$ in $D + L$ has in D a unique singular point (a saddle) and at the same time the index of L with respect to $\mathbf{v}$ is $+1$ ([8]).

5°. Denote

$$w = -k^{-1}(x\psi_x + y\psi_y - \psi) - \frac{a}{2}k^{-2}(\psi_x^2 + \psi_y^2) + \frac{a}{2}.$$

The function* $w = w(\xi, \eta)$ is defined and continuous in $K + C$; by virtue of (2,5), on the boundary of the circle $C:\ \xi^2 + \eta^2 = 1$,

$$w(\xi, \eta)\big|_C = M\left(\theta + \frac{\pi}{2}\right), \tag{2,6}$$

where θ is the polar angle in the (ξ, η)-plane; it satisfies in $K + C$ the equation

$$w_{\xi\xi}w_{\eta\eta} - w_{\xi\eta}^2 = a^2 \tag{2,7}$$

and the inequalities

$$w_{\xi\xi} > 0, \qquad w_\eta > 0; \tag{2,8}$$

moreover, in K the formulas ([9]) hold:

$$w_\xi + a\xi = x, \qquad w_\eta + a\eta = y. \tag{2,9}$$

Corollary 1. A solution of problem I for which L has property E does not exist if

$$a < kG^{-1}b^{-1}.$$

3. Theorem 2. Let the oval Γ be symmetric and elongated along the axis Ox, have only four vertices, and let its radius of curvature $\rho(\beta)$ be an analytic function of β on $[\pi/2, 5\pi/2]$**; then for**

$$a > kG^{-1}\rho_{\min}^{-1}, \tag{3,1}$$

where $\rho_{\min}$ is the minimum radius of curvature, there exists a solution of problem I such that the curve L is symmetric with respect to the axis Ox, the domain D contains the line l of discontinuity of tangential stresses in purely plastic torsion*, L has property E; whereas if $a < k \cdot 2^{-1}G^{-1}\rho_{\min}^{-1}$, then there is no solution of problem I for which L has property E.

The proof of Theorem 2 is based on the existence ([5], $p.132$) of a function $w(\xi, \eta)$, analytic in $K + C$, satisfying in $K + C$ equation (2,7) and inequalities (2,8), and on C condition (2,6). By formulas (2,9) a homeomorphic mapping of $K + C$ onto some domain D with boundary L of the (x, y)-plane is effected. The equation of L—the image of C—can be represented in the form (2,2), where

$$N(\beta) = M(\beta) - w_r(1, \beta - \pi/2) - a,$$

where

$$w_r(1, \theta) \equiv \partial w(r, \theta)/\partial r\big|_{r=1}, \qquad r = \sqrt{\xi^2 + \eta^2}, \qquad \beta = \theta + \pi/2.$$

From the a priori estimate of $w_r(1, \theta)$ and (3,1) it follows that L lies inside Γ.

Corollary 2. Consider the position of the boundary L for different angles α_1, α_2 ($\alpha_1 < \alpha_2$). Let $N_1(\beta)$ correspond to α_1, and $N_2(\beta)$ to α_2; then

$$N_1(\beta) + kG^{-1}(\alpha_1^{-1} - \alpha_2^{-1}) \geq N_2(\beta) \geq N_1(\beta) + k \cdot 2^{-1} G^{-1}(\alpha_1^{-1} - \alpha_2^{-1}).$$

Remark. Theorem 2 and Corollary 2 are also valid for a convex contour Γ_n close to a regular n-gon****

$$x = R\,[f \cos t + (n-1)^{-1} f^{1-n} \cos(n-1)t],$$

$$y = R\,[f \sin t + (n-1)^{-1} f^{1-n} \sin(n-1)t],$$

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* If $w(\xi, \eta)$ is a quadratic polynomial, we obtain the case first considered in work ([10]).

** Such contours include, for example, Lamé curves

$$(xm^{-1})^{2p} + (yn^{-1})^{2p} = 1, \quad p = 1, 2, \dots,$$

for which

$$M(\beta) = \left[(m \sin\beta)^{2p/(2p-1)} + (n \cos\beta)^{2p/(2p-1)}\right]^{(2p-1)/2p}.$$

*** The line l is the segment of the axis Ox joining the centers of curvature of the two points of Γ lying on the axis Ox.

**** See the graph of Γ_n for $n = 3$ and $f = 1.3$ in ([11]), p. 179.

where $f = \sqrt[n]{n-1} + \delta,\ \delta > 0,\ 0 \leq t \leq 2\pi$. In this case the line l consists of n segments drawn from the point of intersection of all axes of symmetry of Γ_n to the centers of curvature of the points of Γ_n corresponding to $t = \frac{2\pi}{n} i,\ i = 0, 1, \dots, (n-1)$, while the curve L is symmetric with respect to the same axes as Γ_n.

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Note: Figure translations are in progress. See original paper for figures.