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Abstract

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THE RIEMANN BOUNDARY-VALUE PROBLEM IN THE SPACE OF GENERALIZED FUNCTIONS AND FABER POLYNOMIALS

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The paper sets forth a theory of generalized functions defined on a basic space consisting of infinitely differentiable functions of points of a sufficiently smooth contour. On the basis of this theory it is possible to solve, in closed form, the Riemann boundary-value problem whose free term is a generalized function, and to interpret the solution as the limiting value of a piecewise analytic function. The formulas giving the solution of the problem can be put in the form that they have in the classical case ([1,2]). The paper continues investigations carried out by various authors in ([3–7]).

§ 1. Let L be a rectifiable Jordan curve. Denote by $w = \Phi(z)$ the function mapping the exterior of the curve L—the domain D^-—onto the exterior of the unit circle $|w| > 1$, under the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$, and let $z = \Psi(w)$ be the function inverse to $w = \Phi(z)$, while $\Phi_n(z)$ are the Faber polynomials for the domain D^+ bounded by the curve L.

Theorem 1. *If L is a closed Jordan curve such that $\Psi'(e^{i\theta})$ satisfies, with respect to θ, the Lipschitz condition*

$$|\Psi'(e^{i\theta_1}) - \Psi'(e^{i\theta_2})| < A|\theta_1 - \theta_2|, \qquad A = \text{const},$$

then every function $\varphi(t)$ of points of the contour L satisfying the Hölder condition

$$|\varphi(t_1) - \varphi(t_2)| < K|t_1 - t_2|^\alpha, \qquad K = \text{const}, \qquad 0 < \alpha \leq 1,$$

can be expanded in the uniformly convergent series

$$\varphi(t) = \sum_{k=0}^{\infty} \varphi_k \Phi_k(t) + \sum_{k=0}^{\infty} \widetilde{\varphi}_k \Psi_k(t), \tag{1}$$

where $\Phi_k(t)$ are the Faber polynomials for the domain D^+, $\Psi_k(t) = \Phi'(t)/\Phi^{k+1}(t)$, and the coefficients φ_k and $\widetilde{\varphi}_k$ are determined by the formulas

$$\varphi_k = \frac{1}{2\pi i}\int_L \varphi(t)\Psi_k(t)\,dt, \qquad \widetilde{\varphi}_k = \frac{1}{2\pi i}\int_L \varphi(t)\Phi_k(t)\,dt.$$

The proof of this theorem is based on the results of S. Ya. Al'per ([8]), who proved the expandability of a broad class of functions analytic in a closed domain in Faber series uniformly convergent in this closed domain.

The formulas for the coefficients follow from the orthogonality of the system of functions $\Phi_k(t)$, $\Psi_k(t)$ on the contour L:

$$\frac{1}{2\pi}\int_L \Phi_k(t)\Phi_n(t)\,dt = 0, \qquad \frac{1}{2\pi i}\int_L \Psi_k(t)\Psi_n(t)\,dt = 0,$$

$$\frac{1}{2\pi i}\int_L \Phi_k(t)\Psi_l(t)\,dt = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

We now consider on L a set S consisting of infinitely differentiable functions $\varphi(t)$. We shall say that a sequence $\{\varphi_k\}$, $\varphi_k \in S$, tends to zero as $k \to \infty$, and write $\lim\limits_{k\to\infty} \varphi_k(t) = 0$ or $\varphi_k(t) \to 0$,

if, for any $s = 0, 1, 2, ...$,

$$\lim_{k\to\infty} \max \left| \frac{d^s \varphi_k}{dt^s} \right| = 0.$$

In what follows we shall call S the **basic space**.

The subspace S^+, by definition, is formed by those functions $\varphi^+ \in S$ which are boundary values of functions analytic inside L. The subspace S^- consists of basic functions that are boundary values of functions analytic outside L and vanishing at infinity. We shall call **generalized functions** (g.f.) linear functionals defined on the space S and having the property that $(f, \varphi_k) \to 0$ if $\varphi_k(t) \to 0$ in the sense indicated above.* If the functional $(f, \varphi^+) = 0$ for $\varphi^+ \in S^+$, then f is called a g.f. of plus type and is denoted by the plus sign (f^+). Similarly, g.f. of minus type (f^-) are introduced.

Theorem 2. *If the contour L is such that $\Psi(e^{i\theta})$ has derivatives of arbitrary order with respect to the variable θ, then every g.f. ν, defined on the basic space S, expands into the series*

$$\nu = \sum_{k=0}^{\infty} \nu_k \Phi_k(t) + \sum_{k=0}^{\infty} \tilde{\nu}_k \Psi_k(t), \tag{2}$$

convergent in the sense of convergence in the space of g.f.; i.e., for every $\varphi(t) \in S$

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \left(\nu - \sum_{k=0}^{m} \nu_k \Phi_k(t) - \sum_{k=0}^{n} \tilde{\nu}_k \Psi_k(t),\ \varphi(t) \right) = 0.$$

Here

$$\nu_k = \frac{1}{2\pi i}(\nu, \Psi_k(t)), \qquad \tilde{\nu}_k = \frac{1}{2\pi i}(\nu, \Phi_k(t)).$$

Corollary. *If $\varphi(t) \in S$, and ν is a g.f. on S, with the contour L satisfying the conditions of Theorem 2, then an analogue of Parseval's equality holds*

$$\frac{1}{2\pi i}(\nu, \varphi) = \sum_{k=0}^{\infty} \nu_k \widetilde{\varphi}_k + \sum_{k=0}^{\infty} \tilde{\nu}_k \varphi_k,$$

where $\nu_k, \widetilde{\varphi}_k, \tilde{\nu}_k, \varphi_k$ are defined as above.

Proof follows directly from Theorems 1 and 2. Let us note that g.f. of plus type, and only they, have an expansion into a series of the form

$$f^+(t) = \sum_{k=0}^{\infty} f_k \Phi_k(t),$$

whereas g.f. of minus type, and only they, have one of the form

$$f^-(t) = \sum_{k=0}^{\infty} \tilde{f}_k \Psi_k(t).$$

Theorem 3. *If the contour L satisfies the conditions of Theorem 1, then the expansion*

$$\frac{1}{t-z} = \sum_{k=0}^{\infty} \Phi_k(z)\Psi_k(t),$$

holds.

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* It is easy to see that this requirement is equivalent to the following formally stronger condition: there exists an r, depending on f, such that from the equalities

$$\lim_{k\to\infty} \max |\varphi_k^{(s)}(t)| = 0, \qquad 0 \le s \le r,$$

there follows the equality

$$\lim_{k\to\infty} (f, \varphi_k) = 0.$$

The proof of the equivalence of these requirements, which define the continuity of the functional, is given in [9], p. 21, for the case of linear functionals on an interval, but it remains valid in our case as well.

uniformly convergent inside D^+ for $t \in L$, and the expansion

$$\frac{1}{t-z} = -\sum_{k=0}^{\infty} \Phi_k(t)\Psi_k(z),$$

uniformly convergent inside D^- for $t \in L$.

The **proof** is based on results obtained in P. K. Suetin’ s paper ([10]).

If the contour L satisfies the conditions of Theorem 2 and ν is a generalized function defined on S, then from Theorem 3 there follow the expansions, uniformly convergent inside the corresponding domains, of “functionals of Cauchy type” into series:

$$\text{for } z \in D^+ : \qquad \frac{1}{2\pi i}\left(\nu, \frac{1}{t-z}\right) = \sum_{k=0}^{\infty} \nu_k \Phi_k(z),$$

$$\text{for } z \in D^- : \qquad \frac{1}{2\pi i}\left(\nu, \frac{1}{t-z}\right) = -\sum_{k=0}^{\infty} \tilde{\nu}_k \Phi_k(z).$$

Let us also point out an interesting representation of the Dirac δ-function in the form of a series:

$$\delta(t-x) = \frac{1}{2\pi i}\left[\sum_{k=0}^{\infty} \Phi_k(x)\Psi_k(t) + \sum_{k=0}^{\infty} \Phi_k(t)\Psi_k(x)\right], \qquad t, x \in L.$$

Theorem 4. If the contour L satisfies the conditions of Theorem 2, then every generalized function of plus type

$$f^+(t) = \sum_{k=0}^{\infty} f_k \Phi_k(t)$$

is analytically continued from the contour into the domain D^+ in the sense that the series

$$\sum_{k=0}^{\infty} f_k \Phi_k(z)$$

converges in D^+. More precisely, the sequence of partial sums of the series

$$\sum_{k=0}^{\infty} f_k \Phi_k(z)$$

converges uniformly on every closed subset lying inside D^+, and on the contour it has as its limit (in the sense of convergence in the space of generalized functions) the generalized function f^+.

The **proof** is based on the representation of the series

$$\sum_{k=0}^{\infty} f_k \Phi_k(z)$$

by means of a "functional of Cauchy type"

$$\sum_{k=0}^{\infty} f_k \Phi_k(z) = \frac{1}{2\pi i}\left(f^+, \frac{1}{t-z}\right).$$

An analogous theorem also holds for generalized functions of minus type.

§ 2. We now proceed to the solution of the Riemann boundary-value problem

$$f^+ = Gf^- + g \tag{3}$$

under the assumption that $G(t) \in S$ on L, and $g(t)$ is a generalized function. The unknowns are the generalized functions $f^+(z)$ and $f^-(z)$, respectively of plus type and of minus type.

Theorem 5. The Riemann problem in the class of generalized functions on S with $\operatorname{ind} G(t) = \chi \geq 0$ is unconditionally solvable. The number of arbitrary constants entering the solution is equal to χ.

Proof. Let $G(t) = X^+(t)/X^-(t)$, where $X(z)$ is the canonical function of the homogeneous problem ([1,2]). Then the boundary condition can be given the form

$$f^+[X^+]^{-1} = f^-[X^-]^{-1} + g[X^+]^{-1}.$$

Since $X^+(t)$ belongs to S and is nonzero on L, $[X^+(t)]^{-1}$ also belongs to S, and, consequently, $g[X^+(t)]^{-1}$ will be a generalized function on S^* and, by Theorem 2, it can be expanded in the series

$$g[X^+]^{-1} = \sum_{k=0}^{\infty} h_k \Phi_k(t) + \sum_{k=0}^{\infty} \tilde{h}_k \Psi_k(t).$$

Applying to the boundary condition, rewritten in the form

$$g[X^+]^{-1} = \sum_{k=0}^{\infty} h_k \Phi_k(t) + \sum_{k=0}^{\infty} \tilde{h}_k \Psi_k(t),$$

Liouville' s theorem for generalized functions ([5]), we obtain for the unknown analytic functions the formulas

$$f^{+}(z) = \left[\frac{1}{2\pi i}\left(g[X^{+}]^{-1}, \frac{1}{t-z}\right) + P_{\varkappa-1}(z)\right] X^{+}(z),$$

$$f^{-}(z) = \left[\frac{1}{2\pi i}\left(g[X^{+}]^{-1}, \frac{1}{t-z}\right) + P_{\varkappa-1}(z)\right] X^{-}(z), \quad (4)$$

where $P_{\varkappa-1}$ is a polynomial of degree $\varkappa - 1$ with arbitrary coefficients.

These formulas differ from the classical ones, derived for the case when $G(t)$ and $g(t)$ satisfy the Hölder condition ([1,2]), in that instead of integrals there are written "Cauchy-type functionals," defined by virtue of the fact that in them the role of the fundamental function is played by

$$\frac{1}{t-z},$$

a function which, for $z \in D^{+}$ or $z \in D^{-}$, belongs to the fundamental space.

Analogously to what was done above, one can establish that in the case of a negative index the problem, generally speaking, has no solutions, and derive the conditions for its solvability.

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* The product of the fundamental function $\Psi(t)$ by a generalized f is defined by the equality

$$(f\Psi, \varphi) = (f, \Psi\varphi) \quad ([11]).$$

Note: Figure translations are in progress. See original paper for figures.