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MATHEMATICAL PHYSICS

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1963

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Abstract

Full Text

MATHEMATICAL PHYSICS

B. M. STEPANOV

ON THE QUESTION OF THE CONSTRUCTION OF THE S -MATRIX

(Presented by Academician N. N. Bogolyubov on 8 II 1963)

In constructing the S -matrix by perturbation theory, one starts from the following expansion in powers of the coupling constant ⁽¹⁾:

$$S = 1 + \int S_1(x) dx + \frac{1}{2!} \int S_2(x_1, x_2) dx_1 dx_2 + \dots$$

First of all there arises the problem of constructing the individual symmetric operators $S_m(x_1, \dots, x_m)$. For their determination physical conditions imposed on the S -matrix were formulated, such as causality, unitarity, Lorentz invariance, etc. Formally all these conditions can be satisfied by putting

$$S_m(x_1, \dots, x_m) = i^m T \{ \mathcal{L}(x_1) \dots \mathcal{L}(x_m) \}.$$

These expressions are very transparent, but they have the unpleasant property that, generally speaking, they lead to the appearance of ultraviolet divergences in momentum space. In this connection a special subtraction method was developed, which is customarily called the R -operation ⁽¹⁾. This method makes it possible to give a rigorous mathematical meaning to the chronological-ordering operation T . In doing so, a preliminary regularization is used in an essential way (in particular, the Lorentz-covariant Pauli-Villars regularization) of the propagators characterizing the propagation of the various particles whose mutual interaction is described by the S -matrix under consideration.

We shall now indicate one possibility for constructing the successive S_m , based on the method of mathematical induction and allowing one to dispense, at least in principle, with a preliminary regularization of the propagators. In this case meaningless infinite expressions do not appear at any stage of the calculations.

Suppose that we have succeeded, for all $m \leq n$, in constructing expressions S_m satisfying the conditions imposed on them and having the form of sums of certain polylocal operators. Then, in particular, for each $m \leq n$ there exist such integers q_m, r_m that for every function $F(x_1, \dots, x_m) \in C(q_m, r_m)$ the linear functional

$$\int S_m(x_1, \dots, x_m) F(x_1, \dots, x_m) dx_1 \dots dx_m. \quad (1)$$

is defined. Among the physical conditions determining S_{n+1} , of particular importance is the causality condition, which is usually written in the form

$$S_{n+1}(y, x_1, \dots, x_n) = R_{n+1}(y, x_1, \dots, x_n),$$

$$\text{if } y \succ x_j \text{ at least for one } j, \quad (2)$$

where it is put

$$R_{n+1}(y, x_1, \dots, x_n) =$$

$$= - \sum_{(0 \leq k \leq n-1)} P\left(\frac{x_1, \dots, x_k}{x_{k+1}, \dots, x_n}\right) S_{k+1}(y, x_1, \dots, x_k) S_{n-k}^+(x_{k+1}, \dots, x_n).$$

First of all, let us give the causality condition (2) a rigorous mathematical meaning. For this purpose we note that, according to N. N. Bogolyubov' s theorem ^(1,2) on multiplication of even-frequency parts of Jordan-Pauli permutation functions, and also thanks to the assumed existence

of the linear functionals (1) for all $m \leq n$, one can always find sufficiently large integers q, r such that the introduced operator R_{n+1} defines a linear functional

$$\int R_{n+1}(x_1, \dots, x_{n+1}) F(x_1, \dots, x_{n+1}) dx_1 \dots dx_{n+1}$$

for all functions $F(x_1, \dots, x_{n+1})$ belonging to the class $C(q, r)$. We single out in $C(q, r)$ the subclass of functions $F_1(x_1, \dots, x_{n+1})$ possessing the property

$$F_1(x_1, \dots, x_{n+1}) = 0, \quad \text{if } x_j > x_1 \text{ for all } j > 1.$$

We define on this subclass of functions the symmetric operator $S_{n+1}(x_1, \dots, x_{n+1})$ by means of the relation

$$\int S_{n+1}(x_1, \dots, x_{n+1}) F_1(x_1, \dots, x_{n+1}) dx_1 \dots dx_{n+1}$$

$$= - \int R_{n+1}(x_1, \dots, x_{n+1}) F_1(x_1, \dots, x_{n+1}) dx_1 \dots dx_{n+1}, \quad (3)$$

which is a strict formulation of the causality condition (2). We shall take this relation as the basis for the subsequent analysis.

Let us consider what consequences follow from the symmetry condition for the operator $S_{n+1}(x_1, \dots, x_{n+1})$. Denote by \mathcal{P} the operator of some permutation of the arguments. Then the symmetry condition may be written as

$$\int S_{n+1} F_1 dx_1 \dots dx_{n+1} = \int (\mathcal{P} S_{n+1}) F_1 dx_1 \dots dx_{n+1}$$

for any permutation \mathcal{P} of the arguments in the operator S_{n+1} . From this condition and from the causality condition (3) it follows that S_{n+1} is also defined on any of the subclasses of $C(q, r)$ of functions $F_i(x_1, \dots, x_{n+1})$, $i = 1, 2, \dots, n+1$, with the properties

$$F_i(x_1, \dots, x_{n+1}) = 0, \quad \text{if } x_j > x_i \text{ for all } j \neq i. \quad (4)$$

Thus S_{n+1} proves to be defined on all functions of the class $C(q, r)$ that can be represented in the form of a sum

$$F(x_1, \dots, x_{n+1}) = \sum_{(i)} F_i(x_1, \dots, x_{n+1}). \quad (5)$$

It is obvious that any such function $F(x_1, \dots, x_{n+1})$ has a zero at the point where all the arguments coincide, $x_1 = x_2 = \dots = x_{n+1}$.

We shall now establish that, conversely, any function in $C(q, r)$ that has a zero of sufficiently high order when all the arguments coincide can be represented in the form of the sum (5).

We first prove an auxiliary, weaker assertion, namely: every function $F_\varepsilon(x_1, \dots, x_{n+1}) \in C(q, r)$ that vanishes in the region

$$|x_2 - x_1| < \frac{\varepsilon}{\sqrt{n}}, \quad |x_3 - x_1| < \frac{\varepsilon}{\sqrt{n}}, \dots, \quad |x_{n+1} - x_1| < \frac{\varepsilon}{\sqrt{n}} \quad (6)$$

is representable in the form of the sum (5), where, of course, the functions F_i also vanish in the region (6).

Construct two infinitely differentiable functions $h(u)$ and $\Delta_\varepsilon(x)$, satisfying the following conditions:

$$h(u) = \begin{cases} \frac{1}{2}, & u = 0, \\ 1, & u \geq 1, \end{cases} \quad h(-u) = 1 - h(u),$$

$$\Delta_\varepsilon(x) = \begin{cases} 0, & |x| \leq \frac{\varepsilon}{3\sqrt{n}}, \\ 1, & |x| \geq \frac{\varepsilon}{\sqrt{n}}, \end{cases} \quad \Delta_\varepsilon(x) = \Delta_\varepsilon(-x).$$

and with their aid define the following functions, $i = 1, 2, \dots, n + 1$:

$$f_\varepsilon^{(i)}(x_1, \dots, x_{n+1}) = \begin{cases} 0, & \text{if all } |x_j - x_i| \leq \frac{\varepsilon}{3\sqrt{n}}, \\ \frac{2}{n+1} \frac{\sum_{(j)} h\left(\sqrt{2} \frac{x_j^0 - x_i^0}{|x_j - x_i|}\right) \Delta_\varepsilon(x_j - x_i)}{\sum_{(j)} \Delta_\varepsilon(x_j - x_i)}, & \text{in the remaining cases.} \end{cases}$$

Then we shall have identically

$$F_\varepsilon(x_1, \dots, x_{n+1}) = \sum_{(i)} f_\varepsilon^{(i)}(x_1, \dots, x_{n+1}) F_\varepsilon(x_1, \dots, x_{n+1}).$$

Since the functions $f_\varepsilon^{(i)}$ possess property (4), the identity established proves our auxiliary assertion.

It is now possible to prove the main assertion. For this purpose take an infinitely differentiable function $g(\tau)$ satisfying the conditions:

$$g(\tau) = \begin{cases} 1, & \tau \leq 1, \\ 0, & \tau \geq 2. \end{cases}$$

Let $F(x_1, \dots, x_{n+1}) \in C(q, r)$ have a zero of sufficiently high order when all arguments coincide, $x_1 = x_2 = \dots = x_{n+1}$. Then the function

$$F_\varepsilon(x_1, \dots, x_{n+1}) = F(x_1, \dots, x_{n+1}) \left\{ 1 - g\left(\frac{\sum_{(i)} |x_i - x_1|^2}{\varepsilon^2}\right) \right\}$$

vanishes in the domain (6). On the other hand, it is easy to see that this function tends to $F(x_1, \dots, x_{n+1})$ as $\varepsilon \rightarrow 0$ uniformly in the whole space x_1, \dots, x_{n+1} .

Thus it has been shown that the symmetric operator S_{n+1} , initially defined on the subclass of functions $F_1(x_1, \dots, x_{n+1})$ from the class $C(q, r)$ by means of the causality condition (3), automatically proves to be defined on all functions $F(x_1, \dots, x_{n+1}) \in C(q, r)$ possessing only a zero of sufficiently high order when all arguments coincide.

Now one may simply invoke the Hahn–Banach theorem³, from which it follows that there exists at least one extension of the linear functional defined by the operator S_{n+1} to all functions of the class $C(q, r)$. At the same time there remains arbitrary a certain quasilocal operator, which may be added to S_{n+1} , and further restrictions on it follow from physical conditions not considered here: unitarity, Lorentz invariance, etc.

The author expresses his gratitude to Academician N. N. Bogolyubov and V. S. Vladimirov for fruitful discussions.

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Academy of Sciences of the USSR

Received
17 I 1963

CITED LITERATURE

¹ N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Moscow, 1957.

² V. S. Vladimirov, *Proceedings of the V. A. Steklov Mathematical Institute of the Academy of Sciences of the USSR*, 60, 101 (1961).

³ N. Dunford, J. T. Schwartz, *Linear Operators*, IL, 1962.

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