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Abstract

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A Generalization of Friedmann Models of the Universe

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At the present time the question of a possible change with time of the so-called gravitational constant [1] is being vigorously discussed, and therefore it is of interest to consider Friedmann models of the Universe generalized to this case. Here we shall consider the comparatively simple case in which $\varkappa = \varkappa(R) = \varkappa(a)$ [2], where R is the scalar curvature, itself a function of time, and a is the "radius of curvature." More general cases of spaces, when $\varkappa$ can be generalized in the form of a tensor [3], will be considered below.

In the classical Friedmann models the "expansion" velocity is a quantity varying in time, which seems somewhat strange, since the velocity of motion of the "boundary" (which is especially evident, for example, in the intermediate quasi-Euclidean model) must be equal to the propagation velocity of the first particle, with rest mass equal to zero, which in turn is equal to the speed of light and therefore must be constant. The solutions of the equations found by Friedmann do not satisfy the conditions of rectilinearity of the first characteristic.

Constancy of the expansion velocity can be achieved only by introducing into the Friedmann model an additional tensor of a hypothetical field t_i^k, or, what is especially clear, if one assumes that $\varkappa = \text{const} \cdot a$; in this case the introduction of the field t_i^k is more justified.

As the initial equations we take

$$R_i^k - \frac{1}{2} R \delta_i^k = \varkappa \left(T_i^k + t_i^k\right) - R \delta_i^k \, \partial \ln \varkappa / \partial \ln g; \tag{1}$$

or

$$R_i^k - \frac{1}{2} R \delta_i^k = \varkappa \left(T_i^k + \bar{t}_i^k\right) + R_i^k \, \partial \ln \varkappa / \partial \ln R. \tag{2}$$

Here T_i^k and $\bar{t}_i^k$ are the energy-momentum tensors of a medium consisting of particles having, respectively, nonzero and zero rest masses; t_i^k is a pseudo-tensor describing particles with rest mass equal to zero. The introduction of t_i^k and $\bar{t}_i^k$, in our opinion, is necessary, since the energy of various fields in the Universe and of neutrinos is commensurable with the energy of matter consisting of ordinary particles with nonzero rest mass.

For an isotropic Universe we shall consider, by a unified method, all three cases of spaces of different metrics. In the proper reference system, since $T_i^k = (p+\varepsilon)u_i u^k + \delta_i^k p$, we find that $T_0^0 = -\varepsilon$; $T_1^1 = T_2^2 = T_3^3 = p$; $T = 3p - \varepsilon$; the components of the tensors t_i^k and $\bar{t}_i^k$ remain to be determined, while it must be borne in mind that all t_α^k and $\bar{t}_\alpha^k$ are identical and that $t = 3t_1^1 + t_0^0$; $\bar{t} = 3\bar{t}_1^1 + \bar{t}_0^0 = 0$. The interval is

$$ds^2 = a^2(\eta)\left[d\eta^2 - d\chi^2 - A^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right],$$

where τ is the proper time; $c\,d\tau = a\,d\eta$; $A = \sin\chi,\ \chi,\ \operatorname{sh}\chi$, respectively for the spaces f_1, f_2, f_3, where f_1 is elliptic space, f_2 Euclidean space, and f_3 hyperbolic space. The parameter a is the radius of curvature of the spaces f_1 and f_3. In the case of Euclidean space f_2, the radius of curvature $a \to \infty$, but in this case by $a = b$ we mean simply the scale factor (the scale of distance changes with time). All parameters are functions of time or of the quantity $a = a(\tau)$.

The calculations give the following components of the tensors (4):

$$R_0^0 = \frac{3a\ddot{a}}{c^2 a^2}, \qquad R_{0\alpha} = 0, \quad \text{where } \dot{a} = \frac{da}{dt};$$

$$R_\alpha^\beta = \frac{\delta_\alpha^\beta}{a^2}\left[2\left(\beta_1 + \frac{\dot{a}^2}{c^2}\right) + \frac{a\ddot{a}}{c^2}\right]; \quad R = R_0^0 + R_\alpha^\alpha = \frac{6}{a^2}\left(\beta_1 + \frac{\dot{a}^2}{c^2} + \frac{a\ddot{a}}{c^2}\right);$$

the parameter $\beta_1 = 1, 0, -1$, respectively, for the spaces f_1, f_2, f_3. The volume of the spaces

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\chi_0} a^3 A^2 \sin\theta\, d\theta\, d\varphi\, d\chi = 4\pi a^3 \int_0^{\chi_0} A^2\, d\chi = 2\pi^2 a^3 \beta_2,$$

where

$$\beta_2 = \frac{1}{\pi}\left(\chi_0 + \frac{1}{2}\sin 2\chi_0\right) = 1 \quad (\chi_0 = \pi),$$

$$\beta_2 = \frac{2}{3\pi} \quad (\chi_0 = 1), \qquad \beta_2 = \frac{1}{\pi}\left(-\frac{1}{\chi_0} + \frac{1}{2}\operatorname{sh} 2\chi_0\right),$$

respectively for the spaces f_1, f_2, f_3; here the quantity χ_0 may to some extent be arbitrary (this fixes the scale and the relation of distance to the total energy of the system). For the spaces f_1 and f_2 it makes sense to fix the quantity χ_0, while for the space f_3 this need not be done.

Since $\chi = \chi(a)$; $-g = a^8 A^4 \sin^2\theta$; $R = \dfrac{6}{a^2}\left(\beta_1 + \dfrac{\dot a^2}{c^2} + \dfrac{a\ddot a}{c^2}\right)$, we have

$$\partial \ln g/\partial \ln a = 8; \qquad \partial \ln R/\partial \ln a = -2\left\{1 - \frac{a\ddot a}{2c^2\left(\beta_1 + \dot a^2/c^2 + a\ddot a/c^2\right)}\right\};$$

therefore the quantity $\partial \ln \varkappa/\partial \ln g$, entering equation (1), may be written in the form $\partial \ln \varkappa/\partial \ln g = \frac{1}{8}\, d\ln\varkappa/d\ln a$.

Similarly we have

$$\frac{\partial \ln \varkappa}{\partial \ln R} = -\frac{1}{2}\frac{d\ln\varkappa}{d\ln a}\left[\frac{1}{1 - a\ddot a/2c^2\left(\beta_1 + \dot a^2/c^2 + a\ddot a/c^2\right)}\right]. \tag{3}$$

In what follows we shall use equations (1). There are three independent equations in all:

$$R_0^0 - \frac{1}{2}R = \varkappa\left(T_0^0 + t_0^0\right) - \frac{1}{8}R\, d\ln\varkappa/d\ln a; \tag{4}$$

$$R = -\varkappa T + \frac{1}{2}R\, d\ln\varkappa/d\ln a; \tag{5}$$

$$R_1^1 - \frac{1}{2}R = \varkappa\left(T_1^1 + t_1^1\right) - \frac{1}{8}R\, d\ln\varkappa/d\ln a. \tag{6}$$

Substituting here $T = 3p - \varepsilon$, $T_0^0 = -\varepsilon$, $T_1^1 = p$ and the values of R_0^0, R_1^1 and R, we write these equations in the form

$$3\left(\beta_1 + \dot a^2/c^2 + a\ddot a/c^2\right)\left(1 - \frac{1}{4}d\ln\varkappa/d\ln a\right) = a^2\varkappa(\varepsilon - t_0^0) + 3a\ddot a/c^2, \tag{7}$$

$$6\left(\beta_1 + \dot a^2/c^2 + a\ddot a/c^2\right)\left(1 - \frac{1}{2}d\ln\varkappa/d\ln a\right) = a^2\varkappa(\varepsilon - 3p); \tag{8}$$

$$\left(\beta_1 + \dot a^2/c^2 + a\ddot a/c^2\right)\left(1 - \frac{3}{4}d\ln\varkappa/d\ln a\right) = -a^2\varkappa(p + t_1^1) - a\ddot a/c^2. \tag{9}$$

Instead of the last equation one may simply write

$$t_1^1 = -\frac{1}{3}t_0^0. \tag{10}$$

Let us now write the law of conservation of the total energy E_0, i.e. the energy of matter and of the gravitational field. Evidently,

$$-\left(T_0^0 + t_0^0\right) V = E_0 = 2\pi^2 a^3 \beta_2 (\varepsilon - t_0^0). \tag{11}$$

Since the equation

$$\varepsilon V = c^2 + pV/(k-1), \tag{12}$$

holds, where for very large $p \sim \varepsilon$, $k = \frac{4}{3}$, while for $p \ll \varepsilon$, k may be arbitrary $(k > 1)$, we may write that

$$\varepsilon - 3p = (c^2/V)\bar{\beta} = \rho^* c^2 = \bar{\beta} E_M / 2\pi^2 a^3 \beta_2 = E_M^* / 2\pi^2 a^3 \beta_2, \tag{13}$$

where $\bar{\beta} = 1 + (pV/c^2(k-1))(4-3k)$, and ρ^* and E_M^* are the density and energy of matter with allowance for the pressure of the medium, while E_M is the energy of matter.

Substituting (11) into (7) and (13) into (8), we arrive at the equations

$$\left(\beta_1 + \dot{a}^2/c^2 + a\ddot{a}/c^2\right)\left(1 - \frac{1}{4} d\ln\varkappa / d\ln a\right) = a\ddot{a}/c^2 + \varkappa E_0 / 6\pi^2 \beta_2 a; \tag{14}$$

$$\left(\beta_1 + \dot{a}^2/c^2 + a\ddot{a}/c^2\right)\left(1 - \frac{1}{2} d\ln\varkappa / d\ln a\right) = \varkappa E_M^* / 12\pi^2 \beta_2 a. \tag{15}$$

Excluding from this

$$\varkappa / a = 3\pi^2 \beta_2 \left(\beta_1 + \dot{a}^2/c^2 - a\ddot{a}/c^2\right) / (E_0 - {}^1/_4 E_{\mathrm{M}}^*),$$

we arrive at the important equation

$${}^3/_4 \frac{(\beta_1 + a^2/c^2)^2}{E_0 - {}^1/_4 E_{\mathrm{M}}^*} \left({}^4/_3 E_0 - E_{\mathrm{M}}^*\right) = \left(\beta_1 + \frac{\dot{a}^2}{c^2}\right)\left(\frac{a\ddot{a}}{c^2} - \frac{a^2\dot{a}}{\dot{a}c^2}\right) - \frac{a^3\dot{a}\ddot{a}}{\dot{a}c^4} + \frac{a^2\dot{a}^2}{c^4}\,\frac{2E_0 - E_{\mathrm{M}}^*}{E_0 - {}^1/_4 E_{\mathrm{M}}^*} - \frac{E_{\mathrm{M}}^*}{E_0 - {}^1/_4 E_{\mathrm{M}}^*}\,\frac{a\ddot{a}}{c^2}\left(\beta_1 \right. \tag{16}$$

The equation

$$\left[\varkappa (T_i^k + t_i^k)\right]_{;k} - \left(\delta_i^k R \frac{\partial \ln\varkappa}{\partial \ln g}\right)_{;k} = \left[\varkappa (T_i^k + t_i^k)\right]_{;k} - {}^1/_8 \frac{\partial}{\partial x_i}\left(R \frac{\partial \ln\varkappa}{\partial \ln g}\right) = 0,$$

as is easy to verify, is satisfied automatically.

Let us now write the equation for the propagation of light rays

$$ds^2 = c^2 d\tau^2 - a^2 d\chi^2 = a^2(d\eta^2 - d\chi^2) = a^2 d(\eta^2 - \chi^2) = 0$$

or

$$\eta = \pm\chi + \text{const}, \tag{17}$$

whence it follows that $c\,dt = a\,d\chi$. Since the expansion of the region occupied by matter must occur with the speed of light, we find that in this case the proper time $\tau = \text{const} = 0$, $d\tau = 0$, therefore at the boundary $\eta = \chi = \chi_0 = \text{const}$. Further, since the distance in the spaces under consideration is $l = a\chi$, the boundary condition $l = l_0 = a\chi_0 = c\tau_0$ must be satisfied, where τ_0 refers to the chosen origin of coordinates (which may be arbitrary); at the same time τ_0 is the time elapsed from the beginning of the expansion or "scattering of space." Hence $a/c = \tau_0/\chi_0$, $\dot{a}/c = 1/\chi_0$. In this case equation (16) takes the simple form

$$^3/_4\,\frac{\beta_1 + 1/\chi_0^2}{E_0 - {}^1/_4 E^*_{\mathrm{M}}}\,({}^4/_3 E_0 - E^*_{\mathrm{M}}) = 0, \tag{18}$$

whence we find that ${}^4/_3 E_0 = E^*_{\mathrm{M}}$. Next we determine

$$\frac{\varkappa}{a} = \frac{9\pi^2\beta_2}{2E_0}\left(\beta_1 + \frac{1}{\chi_0^2}\right) = \frac{6\pi^2\beta_2}{E^*_{\mathrm{M}}}\left(\beta_1 + \frac{1}{\chi_0^2}\right); \qquad \frac{d\ln\varkappa}{d\ln a} = 1, \tag{19}$$

after which it is easy to calculate, on the basis of (11) and (13), that

$$(\varepsilon - 3p)/(\varepsilon - t_0^0) = E^*_{\mathrm{M}}/E_0 = {}^4/_3, \tag{20}$$

whence

$$t_0^0 = -3t_1^1 = {}^1/_4(\varepsilon + 9p). \tag{21}$$

Equations (7) and (8) are now conveniently written in the form

$$t_0^0 = -3t_1^1 = -\varepsilon_g = 3\left[p + \frac{1}{\varkappa a^2}\left(\beta_1 + \frac{1}{\chi_0^2}\right)\right];$$

$$\varepsilon = 3\left[p + \frac{1}{\varkappa a^2}\left(\beta_1 + \frac{1}{\chi_0^2}\right)\right] \tag{22}$$

or

$$-t_0^0 = 3t_1^1 = \varepsilon_g = -\frac{\varepsilon}{4}\,\frac{4p + \dfrac{1}{\varkappa a^2}\left(\beta_1 + \dfrac{1}{\chi_0^2}\right)}{p + \dfrac{1}{\varkappa a^2}\left(\beta_1 + \dfrac{1}{\chi_0^2}\right)} =$$

$$= -\frac{\varepsilon}{4}\left[1 + \frac{3p}{p + \dfrac{1}{\varkappa a^2}\left(1 + \dfrac{1}{\chi_0^2}\right)}\right], \tag{23}$$

where ε_g is the energy density of the gravitational field.

Equation (3) takes the simple form

$$\partial \ln \chi / \partial \ln R = -\frac{1}{2}\, d \ln \chi / d \ln a = -\frac{1}{2}. \tag{24}$$

From equations (1) and (2) we have

$$(\bar{t}_i^k - t_i^k) = -\theta_i^k = \frac{1}{2}\chi\left(R_i^k - \frac{1}{4}\delta_i^k R\right), \tag{25}$$

where $\theta_i^i = \theta = 0$.

The pseudotensor $\bar{t}_i^k$ or θ_i^k may be interpreted as the pseudotensor of the proper gravitational field, and the tensor t_i^k as the tensor of particles having zero rest mass. From (20) we have

$$E_0 = \frac{3}{4}E_M^* = E_M^* + E_g, \tag{26}$$

where $E_g = -\frac{1}{4}E_M^*$ is the energy of the gravitational field. The same result can be obtained from (23), integrating over the whole volume.

The components of the tensor $\bar{t}_i^k$ have the values $-\bar{t}_0^0 = \varepsilon = -3p$; $\bar{t}_1^1 = -p$. Relation (26) may help in explaining the magnitude of the mass and energy of the electron; since $E_{\text{em}} = \frac{3}{4}E_M$, where E_{em} is the electromagnetic energy of the Coulomb field, one may think that, since at the present time $p \to 0$ ($p \ll \varepsilon$), then $E_M^* = E_M$; $E_{\text{em}} = \frac{3}{4}m_0 c^2 = E_M + E_g = E_0$; here $E_M = m_0 c^2$, where m_0 is the mass of the electron field. (Let us recall that our results apply in the proper reference system; this allows m_0 to be interpreted as the rest mass.)

In the case when $\chi = \text{const}$ under the condition that $\dot{a} = \text{const}$, using equations (14) and (15), we find that

$$E_0 = \frac{E_{\text{em}}}{2} = \frac{E^*_M}{2} = \frac{m_0 c^2}{2}.$$

These relations are unlikely to have meaning, since they contradict the data of electrodynamics. In this case $t^0_0 = -3t^1_1 = \dfrac{\varepsilon + 3p}{2}$.

The velocity of recession of objects from one another is $v = \partial(a\bar{\chi})/\partial\tau = \bar{\chi}\,\partial a/\partial\tau = \bar{\chi}\dot{a} = l\dot{a}/a = lh$, where $l = a\bar{\chi}$ is the distance, and $\bar{\chi}$ is the (Lagrangian) coordinate of the object receding relative to the observer chosen as the origin of coordinates; τ is the time elapsing for the observer. The quantity $h = \dot{a}/a$ is called the Hubble boundary; it is measured by observations of nebulae. Since $\dot{a}/c = 1/\chi_0$, $v/c = \bar{\chi}/\chi_0 = lh/c$. This quantity is always less than unity (at the boundary of the region it is equal to unity).

Let us now write equation (8) in the form $3(\beta_1/a^2 + \dot{a}^2/a^2c^2) = \chi(\varepsilon - 3p) = \chi c^2\rho^* = 3(\beta_1/a^2 + h^2/c^2)$, whence

$$\beta_1 c^2/a^2 + h^2 = (c^2/a^2)(\beta_1 + 1/\chi_0^2) = 8\pi G\rho^*/3. \tag{27}$$

From observational data it is possible to determine the quantities h and S^*, which will first of all help to estimate the sign of the term $\beta_1 c^2/a^2$, i.e. the type of space, and also the quantity l_0, the size of space.

The investigation of anisotropic spaces with $\chi \neq$ const presents no difficulty; however, in this case, as it seems to us, there is no sense in rejecting the singularity at $T = 0$ (where T is "world time"), since all observational data indicate that it must occur.

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Note: Figure translations are in progress. See original paper for figures.