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Abstract

Full Text

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ANALYSIS AND SYNTHESIS OF RINGS OF TYPE C

(Presented by Academician P. S. Aleksandrov on 4 X 1962)

Let X be a compact group, and let R be a complete normed ring of continuous real-valued functions on X (with pointwise multiplication). The ring R is assumed to be “homogeneous,” i.e., together with each function f it contains all its translates $T^x f$, and the mapping $x \rightarrow T^x$ is a continuous representation of the group X on R . It is also assumed that the ring R is “regular,” i.e., for any $x, y \in X$, $x \neq y$, there exists a function $f \in R$ that is equal to zero in a neighborhood of x and equal to 1 at the point y .

If $\|f\|$ is the norm given on R , then the **growth norm** at the point x will be denoted by $\|f\|^x = \inf\{\|g\| : g = f \text{ in a neighborhood of } x\}$. Among the closed primary ideals contained in the maximal ideal $M^x = \{f : f(x) = 0\}$, the ideal $I^x = \{f : \|f\|^x = 0\}$ is the smallest; it is called the **minimal ideal** at the point x .

One says that R is a **ring of type C** if the norm $\|f\|$ is equivalent to the norm $\sup_x \|f\|^x$. An equivalent formulation is the following.

In a regular homogeneous ring R put $L^x = R/I^x$. The ring L^x is local, with the unique maximal ideal $N^x = M^x/I^x$, and all the rings L^x are canonically isomorphic to the ring L^0 (this isomorphism is effected by means of translation). Let $C(X, L^0)$ be the ring of continuous L^0 -valued functions on X ; for each function $f \in R$ define the function $f^b \in C(X, L^0)$ by setting $f^b(x) = f/I^x = T^x f/I^0$ (where, for example, by f/I^x is denoted the image of f in R/I^x). Then R is a ring of type C if and only if the mapping $f \rightarrow f^b$ takes R onto a closed subring of the ring $C(X, L^0)$.

Rings of type C were first considered by G. E. Shilov. He established the following basic result on analysis of type C (see ⁽¹⁾); Shilov's proof is given for the case of a commutative group X , but his method carries over without difficulty to the case of an arbitrary compact group X).

Theorem. *Let R be a regular ring of type C , and let P be the ring of trigonometric polynomials (i.e., such functions p on X that only a finite number of translates $T^x p$ are linearly independent). Then R contains P and is isomorphic to the closure in $C(X, L^0)$ of the ring P^b .*

The synthesis of type C is the process inverse to the one described above. The starting point in it is the specification of a local ring L (normed, complete) with a unique maximal ideal N and of such an isomorphism b of the ring P into $C(X, L)$ that $f^b(x) = (T^x f)^b(0)$. Denoting by S the closure of P^b in $C(X, L)$, suppose that every element of the ring L occurs as the value $F(0)$ at zero of some function $F \in S$. Denote by \natural the natural homomorphism of the ring $C(X, L)$ onto $C(X)$, defined by factorization in the ring L of the values of functions modulo the ideal N ; suppose that $f^{\natural} = f$ for all polynomials $f \in P$. The ring S of L -valued functions and the corresponding ring S^{\natural} of real-valued functions thus turn out to be synthesized from the ring L and the isomor-

homomorphism b . But in order to understand this construction properly, we must answer several important questions.

Question 1. Is the mapping $S \rightarrow S^{\natural}$ one-to-one?

Question 2. Is the ring S^{\natural} regular?

Question 3. If the ring S^{\natural} is regular, is it a ring of type C ?

G. E. Shilov answered questions 1 and 3 in the affirmative. To question 2 he answered in the negative and gave certain sufficient conditions for regularity.

In the present note we wish to point out one more important question.

Question 4. Let S^{\natural} be a ring of type C , and let I^0 be the minimal ideal at zero; are the local rings S^{\natural}/I^0 and L isomorphic?

The answer is negative. Indeed, there always exists a natural homomorphism $S^{\natural}/I^0 \rightarrow L$, but, as we shall see, this homomorphism may be proper. Thus we shall show, in particular, that one and the same ring S^{\natural} can be synthesized from two different local rings L_1 and L_2 .

The example giving an answer to question 4 is based on the properties of elliptic differential operators. Let X be an n -dimensional torus, and consider only rings of strict type C , i.e., such that R/I^0 is finite-dimensional. A ring R is of strict type C if and only if it contains the ring R_{∞} of all infinitely differentiable functions. In such rings it is convenient to regard the ring R_{∞} as playing the same role as the ring P in the general case. It is not hard to show that the answers to questions 1, 3, 4 do not change if P is replaced by R_{∞} . Question 2 now, of course, has an affirmative answer.

Let us first consider analysis of type C . The mapping $f \rightarrow f^{\natural}(x)$ of the ring R onto L^x is determined by its restriction to the dense subring R_{∞} . Since L^x is finite-dimensional, the image of R_{∞} is all of L^x . It can be shown that the R_{∞} -ideal

$$I_{\infty}^x = \left\{ f \in R_{\infty} : \text{all } \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} f(x) = 0 \right\}$$

is contained in the R -ideal I^x ; consequently, there exists a natural homomorphism $L_\infty^x = R_\infty/I_\infty^x$ onto L^x . But L_∞^x is isomorphic to the ring of all formal power series, and the homomorphism $R_\infty \rightarrow L_\infty^x$ assigns to a function f its Taylor expansion

$$\sum \frac{f^{(\alpha)}(x)}{\alpha!} t^\alpha$$

(here, for n -fold series, we use the abbreviated notation). The homomorphism $L_\infty^x \rightarrow L^x$ is reduction of a series modulo some ideal I_∞^x of finite codimension. Thus, if I_∞^x is of finite codimension, one can find m differential operators with constant coefficients D_1, \dots, D_m such that the image $f^\sharp(x)$ of a function f in L^x is equal to zero if and only if $D_1 f(x) = \dots = D_m f(x) = 0$. The norm of type C will then be equivalent to the norm

$$\sup\{|D_1 f| + \dots + |D_m f|\}.$$

Conversely, for synthesis of type C we choose an arbitrary ideal of finite codimension in the abstract ring L_∞ of formal power series and complete R_∞ in the corresponding norm. By Shilov's main theorem, if first, from the ring R , by analysis of type C , we construct the ideal I_∞ , and then, by synthesis of type C , obtain a new ring R' , then $R = R'$. But if we begin with the ideal I_∞ , construct by synthesis the ring R , and then by analysis obtain a new ideal I'_∞ , we know only that $I'_\infty \subseteq I_\infty$. Question 4 is equivalent to the question whether the ideals I'_∞ and I_∞ coincide, or, in other words, whether every ideal I_∞ in the ring of formal power series can be

obtained by analysis of type C from a certain corresponding ring R of type C . Next we give a counterexample. On the two-dimensional torus consider the norm

$$\sup \left\{ |f| + \sum_{i=1}^3 \left| \frac{\partial^i f}{\partial x^i} \right| + \sum_{i=1}^3 \left| \frac{\partial^i f}{\partial y^i} \right| + \left| \frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4} \right| \right\},$$

corresponding to the ideal generated by the polynomials xy , $x^4 - y^4$ and by all monomials of degree five. Carrying out first the synthesis and then the analysis, we obtain the norm

$$\sup \left\{ |f| + \sum_{1 \leq i+j \leq 3} \left| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right| + \left| \frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4} \right| \right\},$$

corresponding to the ideal in the ring of formal power series generated by the polynomials $x^4 - y^4$, xy^3 , x^2y^2 , x^3y and by all monomials of degree five. To

prove the equivalence of these formally different norms, we turn to the following theorem.

Theorem. Let p be a homogeneous polynomial in n variables, equal to zero only at the origin. Let

$$p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) = p\left(\frac{\partial}{\partial x}\right)$$

be the elliptic operator defined by the polynomial p . Then for any differential operator with constant coefficients $q\left(\frac{\partial}{\partial x}\right)$ of order strictly less than the order of the polynomial p , there exists an $\varepsilon > 0$ such that for all $f \in R_\infty$ the inequality

$$\varepsilon \sup \left| q\left(\frac{\partial}{\partial x}\right) f \right| \leq \sup \left\{ |f| + \left| p\left(\frac{\partial}{\partial x}\right) f \right| \right\}.$$

holds.

The equivalence of the norms already allows us to answer question 4. But in order to verify the requirement that the second norm really arises as a result of analysis of type C , we must show that there is no third equivalent norm formally stronger than the second. For this we use the following theorem.

Theorem. Let $p\left(\frac{\partial}{\partial x}\right)$ and $q\left(\frac{\partial}{\partial x}\right)$ be differential operators with constant coefficients of orders p and q , respectively, $p \leq q$, and let the polynomial q not be a multiple of the polynomial p . Then for any $\varepsilon > 0$ there exists a function $f \in R_\infty$ such that

$$\varepsilon \sup \left| q\left(\frac{\partial}{\partial x}\right) f \right| > \sup \left\{ |f| + \left| p\left(\frac{\partial}{\partial x}\right) f \right| \right\}.$$

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References

1. G. E. Shilov, *Uspekhi Mat. Nauk*, **6**, No. 1 (41), 91 (1951).

Note: Figure translations are in progress. See original paper for figures.

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