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1963

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Abstract

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ON THE HYDRODYNAMICS OF A ROTATING BOSE SYSTEM BELOW THE CONDENSATION POINT

(Presented by Academician N. N. Bogolyubov, 24 V 1963)

As a number of experiments have shown, vortex filaments appear in rotating helium II; their existence was first proposed by Onsager and Feynman ([1,2]). The corresponding system of hydrodynamic equations was proposed by Hall and Vinen ([3]) and, in a more general form, in ([4]), using a phenomenological approach. In the present work the system of hydrodynamic equations is obtained from microscopic theory, which makes it possible to obtain an expression for the mutual-friction force of the normal and superfluid components in terms of certain correlation functions.

In the work of N. N. Bogolyubov ([5]) it was shown that the equations of the two-fluid hydrodynamics of L. D. Landau ([11]) for an arbitrary Bose system follow from the assumption of a macroscopically large number of particles in the condensate. Here an analogous approach is used, taking into account the changes introduced by the presence of singularities associated with vortex filaments.

For a rotating Bose system in equilibrium it can be shown that the phase S of the condensate wave function $\varphi(x) = a(x)e^{iS(x)}$ cannot be a single-valued function of x. Solutions for a model of a weakly nonideal Bose gas near singularities were found in ([7,8]). The corresponding singularity has the form of a vortex filament, so that near the singularity $S = m\vartheta$, where ϑ is the polar angle with respect to the axis of the vortex filament and m is an integer.

In what follows we shall assume that the phase S has special lines—vortex filaments parallel to the axis of rotation—whose circuit gives an increment 2π (the choice $m = 1$ is thermodynamically the most favorable). We consider plane motion (in the plane perpendicular to the axis of rotation) such that an extremely large number of such singularities is contained in an area with linear dimension of the order of the characteristic size for the mean motion. We shall be interested in equations for mean quantities slowly varying in space and time. The average over the positions of the special lines may be understood as an average over their initial positions.

Let us introduce, following ([9]), the condensate function $\varphi(x)$, putting for the second-quantization operators $\psi(x) = \varphi(x) + \psi_1(x)$, $\psi^+(x) = \varphi^*(x) + \psi_1^+(x)$, where ψ_1, ψ_1^+ are operators, and φ, φ^* are c-numbers.

The corresponding equations of motion for φ, φ^* and ψ_1, ψ_1^+ are obtained from the equations of motion for ψ, ψ^+. Setting $a^* = ae^{iS}$, we obtain

$$-\hbar\frac{\partial S}{\partial t} = \frac{\hbar^2}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\Delta a}{a} + \lambda + \int \Phi(x-x')\,\mathrm{Re}\,\frac{\langle\psi^+(x')\psi(x)\psi(x)\rangle e^{-iS(x)}}{a(x)}\,dx', \tag{1}$$

$$\hbar\frac{\partial a}{\partial t} = -\frac{\hbar^2}{2m}\,[2\nabla a\cdot\nabla S + a\Delta S] + \int \Phi(x-x')\,\mathrm{Im}\langle\psi^+(x')\psi(x')\psi(x)\rangle e^{iS(x)}\,dx'. \tag{2}$$

Here $\Phi(x - x')$ is the potential of pair interaction of Bose particles.

Analysis of the singularity near the L-th vortex filament shows that

$$\vec{\nabla} S = \vec{\nabla}\widetilde{S} + \nabla S_L, \qquad \vec{\nabla} S_L = \frac{\mathbf{n}_\vartheta}{|\mathbf{x} - \mathbf{x}_L|}, \quad a = a_1|\mathbf{x} - \mathbf{x}_L| + \cdots,$$

$$\frac{d\mathbf{x}_L}{dt} = \mathbf{v}_L = \frac{\hbar}{m}\,\alpha\,\vec{\nabla}\widetilde{S}\big|_{x=\mathbf{x}_L}, \tag{3}$$

where a_1, α are functions only of time; $\vec{\nabla}\widetilde{S}$ is the part of $\vec{\nabla} S$ that is regular near the filament located at $\mathbf{x}_L$; $\mathbf{n}_\vartheta$ is the unit vector in the direction of rotation at the point $\mathbf{x}$.

In the absence of filaments, the superfluid velocity $\mathbf{v}_s$ is defined as the vector $\frac{\hbar}{m}\vec{\nabla} S$. If the quantities S and a vary slowly in space and time, then the usual hydrodynamic equation for $\mathbf{v}_s$ is obtained directly by applying the operation $\vec{\nabla}$ to equation (1).

In the presence of filaments we shall call the averaged quantity $\frac{\hbar}{m}\overline{\vec{\nabla} S}$ the superfluid velocity, where the bar denotes averaging over different configurations of the filaments.

When taking the configuration average of the gradient of equation (1), a certain caution must be observed, since the integral of the right-hand side with respect to d^2x_L does not exist because of the presence of singularities in $\vec{\nabla} S$. Instead of directly averaging this equation, it is convenient to integrate it over t from t to $t + \delta t$ and to consider the average of the circulation of the resulting expression along some closed contour $\mathcal{L}$. If the singularities are separated out, the regular part gives zero. The integrals containing the singular part are evaluated directly. The contour $\mathcal{L}$ is assumed to be sufficiently large so that it contains a large number of singularities, but the averaged quantities vary little along its extent. Neglecting small terms of order δt (δt being much smaller than the characteristic time of variation of the averaged quantities), we obtain

$$\frac{\partial}{\partial t}\operatorname{rot}\mathbf{v}_s = \operatorname{rot}[\mathbf{v}_L \operatorname{rot}\mathbf{v}_s], \qquad |\operatorname{rot}\mathbf{v}_s| = \frac{2\pi\hbar}{m}n(x), \tag{4}$$

where $n(x)$ is the mean density of filaments per unit area.

Analogously, one may consider the mean flux of the gradient of equation (1) through some fixed cylindrical surface Σ. Separating out the singularities and carrying out the calculation of the corresponding integrals, we find

$$-\frac{\partial}{\partial t}\operatorname{div}\mathbf{v}_s = \operatorname{div}\operatorname{grad}\left\{\frac{\hbar^2}{2m^2}\overline{(\vec{\nabla}S)^2} - \frac{\hbar^2}{2m^2}\frac{\Delta a}{a} + \lambda + \right. \tag{5}$$

$$\left. + \operatorname{Re}\int \Phi(x-x')e^{-iS(x)}\frac{1}{a(x)}\langle\psi^+(x')\psi(x')\psi(x)\rangle\,dx'\right\} - \operatorname{div}[\mathbf{v}_L \operatorname{rot}\mathbf{v}_s].$$

If $\mathbf{v}_s$ varies slowly in space, then allowance for the presence of filaments for the average in braces amounts to taking into account small higher derivatives. In the lowest approximation this average may be calculated as the average in the absence of filaments (with this approach the various small terms obtained phenomenologically in (4) disappear). Finally, from (3) and (4) we obtain the equation for the superfluid velocity

$$\frac{\partial \mathbf{v}_s}{\partial t} = -\operatorname{grad}\left(\frac{v_s^2}{2} + \mu\right) + [\mathbf{v}_L \operatorname{rot}\mathbf{v}_s], \tag{6}$$

where μ is the chemical potential for the local equilibrium state.

The remaining equations for the total density ρ, the momentum density $\rho\mathbf{v}$, and the energy E retain, in the approximation under consideration, the same form as in the absence of filaments (see $(^{11,6})$). This is connected with the fact that in them $\vec{\nabla}S$ enters in the combination $a\vec{\nabla}S$, which, according to (3), has no singularities.

To close the hydrodynamic equations it is necessary to express $\mathbf{v}_L$ in terms of the remaining hydrodynamic quantities. In the lowest approximation the quantity $\mathbf{v}_L$ can be calculated from the problem of a single filament, since taking into account the influence of the other filaments leads to a dependence of $\mathbf{v}_L$ on $\operatorname{rot}\mathbf{v}_s$. For the same reason $\mathbf{v}_L$ can be determined from a stationary problem with homogeneous conditions at infinity.

The equation for φ will in this case have the form

$$-\frac{\hbar^2}{2m}\Delta\varphi + i\hbar(\mathbf{v}_L - \mathbf{v}_s)\vec{\nabla}\varphi - \lambda\varphi + \int \Phi(x-x')\langle\psi^+(x')\psi(x')\psi(x)\rangle\,dx' = 0, \tag{7}$$

where φ must have the prescribed singularity (3) at zero ($x_L = 0$), and, moreover, $\lim_{x\to\infty} \nabla\varphi = 0$, $\lim_{x\to\infty} |\varphi| = a_0$, while all averages of the type $\langle\psi_1^+(x)\psi_1(x')\rangle$, etc., must tend as $x \to \infty$ to their equilibrium values. The brackets in the present case denote an average over the thermodynamically equilibrium state at infinity. The quantity $\mathbf{v}_L$ plays in this problem the role of an eigenvalue.

Integrating a certain identity that follows from (7) and expresses the conservation law of the superfluid momentum over a rectangle with its short side parallel to $\mathbf{v}_L - \mathbf{v}_s$, and then passing to the limit in such a way that the ratio of the sides tends to zero and the lengths of the sides to infinity, one obtains the equality

$$2\pi n\hbar[(\mathbf{v}_L - \mathbf{v}_s)\vec{\nu}] = \frac{n}{a_0^2}\mathbf{F}. \tag{8}$$

Here $\vec{\nu}$ is the unit circulation vector. Equation (8) coincides with the corresponding equation in the work (4). The braking force $\mathbf{F}$ then has the form

$$\mathbf{F} = 2\,\mathrm{Re}\int [X(x) - X(\infty)]\,\varphi\vec{\nabla}\varphi^*\,dx, \tag{9}$$

where

$$X(x) = \int \Phi(x-y)\,\frac{\langle\psi^+(y)\psi(y)\psi(x)\rangle - |\varphi(y)|^2\varphi(x)}{\varphi(x)}\,dy.$$

This expression is exact in the sense that it does not depend on a special model of the Bose system. However, finding the corresponding averages represents an extremely complicated many-particle problem. In the case of the model of a weakly interacting Bose gas (6), the expression for $\mathbf{F}$ is substantially simplified:

$$\mathbf{F} = \iint \Phi(x-y)\langle\psi_1^+(y)\psi_1(y)\rangle\vec{\nabla}a^2(x)\,dx\,dy+ \tag{10}$$

$$+\,2\,\mathrm{Re}\int \Phi(x-y)\,[\varphi^*(y)\langle\psi_1(y)\psi_1(x)\rangle + \varphi(y)\langle\psi_1^+(y)\psi_1(x)\rangle]\,\vec{\nabla}\varphi^*(x)\,dy\,dx,$$

where one may assume that $\varphi(x)$ does not depend on ψ_1, ψ_1^+ and is determined as the solution of (7) with $\psi_1^+ = \psi_1 = v_L \equiv 0$, since terms containing ψ_1, ψ_1^+ in powers higher than the second may be neglected. The corresponding solution was studied in (8,9). To determine $\langle\psi_1^+(x)\psi_1(y)\rangle$ and $\langle\psi_1(y)\psi_1(x)\rangle$ we obtain linear equations.

Expression (10) can be transformed into another form. For this purpose it is convenient to introduce the operators $\psi_1' = e^{-iS}\psi_1$, $\psi_1^{+\prime} = e^{iS}\psi_1^+$ and combine them into one vector $\vec{\psi}_1' = (\psi_1', \psi_1^{+\prime})$ so that the system of equations for them takes the form

$$i\hbar \frac{\partial \vec{\psi}_1'}{\partial t} = \hat{\mathscr{H}}_0 \vec{\psi}_1' + \hat{V} \vec{\psi}_1', \tag{11}$$

where $\hat{\mathscr{H}}_0$ is the homogeneous matrix operator corresponding to $\varphi = a_0$, and $\hat{V}$ is the perturbation operator, taking into account the inhomogeneity of a and S, which vanishes as $x \to \infty$. It is convenient to write the operators $\psi_1^{+\prime}, \psi_1'$ with the aid of a special system of eigenfunctions satisfying

to the equation

$$\vec{\psi}_{\mathbf{k},E_k} = \mathbf{f}_{\mathbf{k},E_k} + \frac{1}{E_k - \hat{\mathscr{H}}_0 + i\varepsilon} \hat{V} \vec{\psi}_{\mathbf{k},E_k}, \tag{12}$$

where $\mathbf{f}_{\mathbf{k},E_k} = (u_k,\, v_k) e^{i\mathbf{kx}}$ is the solution of (11) for $\hat{V} \equiv 0$, describing a freely propagating excitation with wave vector $\mathbf{k}$ and Bogoliubov spectrum $E_k > 0$; u_k, v_k are the known parameters of the corresponding canonical transformation. Since $\vec{\psi}_{\mathbf{k},E_k}$ for $\mathbf{x} \to \infty$ coincide with $\mathbf{f}_{\mathbf{k},E_k}$, the averages of the corresponding Bose operators $\langle c_{\mathbf{k}}^+ c_{\mathbf{k}} \rangle = n(k)$ give the occupation numbers of the excitations incident on the filament. The system of functions (12) must, generally speaking, be supplemented by functions $\vec{\psi}_{\alpha,E_\alpha}$, describing internal oscillations of the filament. Writing the operators ψ_1, ψ_1^+ in terms of the operators $c_{\mathbf{k}}$ and $c_{\mathbf{k}}^+$ and using equation (12), one can show that

$$\begin{aligned} \mathbf{F} = 2\pi \int n(k)(\mathbf{k} - \vec{\varkappa}) \left| \langle \mathbf{f}_{\vec{\varkappa},E_\varkappa} \hat{g} \hat{V} \vec{\psi}_{\mathbf{k},E_k} \rangle \right|^2 \delta(E_k - E_\varkappa) \frac{d^3k}{(2\pi)^3} \frac{d^3\varkappa}{(2\pi)^3} \\ - 2\pi \frac{\hbar}{m} \int n(k) \left[(\mathbf{p}(\mathbf{k}) - \mathbf{p}^0(\mathbf{k}))\mathbf{v} \right] \frac{d^3k}{(2\pi)^3} - \frac{\pi\hbar^2}{2m} \int n(k) \mathbf{R}(\mathbf{k}) \frac{d^3k}{(2\pi)^3}, \end{aligned} \tag{13}$$

where

$$p_j(\mathbf{k}) = \frac{1}{2i\hbar} \left(\frac{\partial \vec{\Psi}_{\mathbf{k}}^*}{\partial x_j} \vec{\Psi}_{\mathbf{k}} - \vec{\Psi}_{\mathbf{k}}^* \frac{\partial \vec{\Psi}_{\mathbf{k}}}{\partial x_j} \right) \Bigg|_{x=0}; \qquad \vec{\Psi}_{\mathbf{k}} = (\psi_{\mathbf{k},E_k} e^{iS},\, \psi_{\mathbf{k},E_k}^+ e^{-iS});$$

$$\mathbf{R}(\mathbf{k}) = \vec{\nabla} \left\{ \vec{\Psi}_{\mathbf{k}E_k}^*(x) \vec{\Psi}_{\mathbf{k},E_k}(x) \right\} \Big|_{x=0};$$

$\mathbf{p}^0$ is calculated analogously to $\mathbf{p}$, but with $\vec{\psi}_{\mathbf{k},E_k}$ replaced by $f_{\mathbf{k},E_k}$;

$$\hat{V}\vec{\psi} = -\frac{i\hbar^2}{m}(\vec{\nabla}S\cdot\vec{\nabla})\vec{\psi} + \frac{\hbar^2}{2m}\frac{\Delta a}{a}\hat{g}\vec{\psi} + \int \Phi(x-y)\,[a(x)a(y)-a_0^2]\,\hat{\delta}\vec{\psi}(y)\,dy;$$

$$\hat{g} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\delta} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

The appearance of the second term in formula (13) is connected with the singularity of $\vec{\nabla}S$ at zero. If $\mathbf{F}$ is calculated in first order of perturbation theory, considering the term $-i\frac{\hbar^2}{m}\vec{\nabla}S\cdot\vec{\nabla}$ as a small correction in the equation for $\vec{\psi}_{\mathbf{k},E_k}$, and if the terms taking account of the variation of a are neglected, which is valid for excitations with small k, then we obtain

$$\mathbf{F} = \mathbf{u}\frac{\hbar^5}{m^2}\pi\int(k^2-\varkappa^2)\frac{k_x(k_y-\varkappa_y)(k_y\varkappa_x-\varkappa_y k_x)}{|\mathbf{k}-\vec{\varkappa}|^4}\left.\frac{\partial n(E)}{\partial E}\right|_{E_k}(u_k u_\varkappa - v_k v_\varkappa)^2$$
$$\times\,\delta(E_k-E_\varkappa)\,\delta(k_z-\varkappa_z)\frac{d^3k}{(2\pi)^3}d^3\varkappa,$$

where $\mathbf{u}$ is the mean velocity of the excitations far from the filament. Since $|\mathbf{k}|$ for small k is a single-valued function of E_k, in this approximation $\mathbf{F} = 0$. Thus, the result obtained differs substantially from the result of the calculation of the braking force for phonons carried out in work [10] on the basis of hydrodynamic representations in the same approximation. We note that this calculation corresponds only to the first term of formula (13).

In conclusion, I express my gratitude to N. N. Bogolyubov for valuable advice in carrying out the present work, and also to S. V. Tyablikov, D. N. Zubarev, and Yu. A. Tserkovnikov for useful discussion.

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Academy of Sciences of the USSR

Received
8 V 1963

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Note: Figure translations are in progress. See original paper for figures.

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