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Abstract

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ON A GENERALIZED CAUCHY FORMULA FOR AN ELLIPTIC SYSTEM OF FIRST ORDER

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1. Let a certain elliptic system of first order in the plane be written in the form of a single complex equation

$$\frac{\partial f}{\partial \bar z} = Af + B\bar f, \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad z \in G. \tag{1}$$

Equations of this type have been studied in the works of I. N. Vekua (see ([1],[2])), where two different approaches are proposed. If $A, B \in L_p(G)$, $p > 2$, then the method of the book ([1]), based on integral representations of solutions, is applicable. If A and B are analytic in x and y, then the method of the paper ([2]), based on analytic continuation of solutions into the domain of complex values of the arguments, is applicable. These two methods complement one another; the special value of the second method appears when A, B are analytic but do not belong to $L_p(G)$, $p > 2$. One such special case (constant coefficients in an unbounded domain) is studied in the present note.

2. Assuming A and B analytic in x, y, continuing them into the domain of the arguments $z = x + iy$, $\zeta = x - iy$, and introducing the new function

$$F = f \exp\left\{-\int A(z, \bar z)\, d\bar z\right\},$$

we obtain

$$\frac{\partial F}{\partial \bar z} = \lambda \bar F, \qquad \lambda = B \exp\left\{-2i\ \mathrm{Im}\int A(z, \bar z)\, d\bar z\right\}. \tag{2}$$

We shall assume that the coefficient λ ($\lambda \neq 0$) in this equation is a constant (in general, complex-valued) quantity on the whole plane. The study of equation (2) is based on its connection with the equation

$$\frac{1}{4}\Delta F = \frac{\partial^2 F}{\partial z\, \partial \bar z} = |\lambda|^2 F, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{3}$$

discovered by I. N. Vekua. Namely, if F is a solution of equation (2), then applying the operator $\partial/\partial z$ with repeated use of equation (2) gives the identity (3). Conversely, if F_1 is a general solution of equation (3), then the formula

$$F = F_1 - \frac{1}{\lambda}\frac{\partial \bar{F}_1}{\partial z}$$

gives the general solution of equation (2) (see ([1]), Ch. 3, § 9).

3. Proceeding from this, we first construct two fundamental solutions of equation (2), linearly independent over the field of real numbers. The general solution of equation (3) depending only on r, where $r = |z - z_0|$, z_0 is an arbitrarily fixed point, has the form

$$aH_0^{(1)}(2i|\lambda|r) + bH_0^{(2)}(2i|\lambda|r),$$

where $H_0^{(1)}, H_0^{(2)}$ are Hankel functions (see, for example, ([3]), p. 108), and a and b are arbitrary constants. Let

$$\begin{aligned} \omega_1 &= a_1 H_0^{(1)}(2i|\lambda|r) + b_1 H_0^{(2)}(2i|\lambda|r), \\ \omega_2 &= a_2 H_0^{(1)}(2i|\lambda|r) + b_2 H_0^{(2)}(2i|\lambda|r). \end{aligned} \tag{4}$$

Then the functions

$$X_1(z, z_0) \equiv \omega_1 - \frac{1}{\bar{\lambda}}\frac{\partial \overline{\omega_1}}{\partial z}, \qquad X_2(z, z_0) \equiv \omega_2 - \frac{1}{\bar{\lambda}}\frac{\partial \overline{\omega_2}}{\partial z} \tag{5}$$

will be solutions of equation (2).

We shall choose the coefficients in formulas (4) in accordance with the following requirements. Following ([1]), p. 179, consider the functions

$$\begin{aligned} \Omega_1(z, \zeta) &= X_1(z, \zeta) + iX_2(z, \zeta), \\ \Omega_2(z, \zeta) &= X_1(z, \zeta) - iX_2(z, \zeta) \end{aligned} \tag{6}$$

and require that, for $z = \zeta$, the function Ω_1 have a pole of the first order with residue 1, while the function Ω_2 have a singularity no worse than logarithmic.

Taking into account that in a neighborhood of $z = 0$ we have

$$H_0^{(k)}(z) = \frac{(-1)^{k+1}2i}{\pi}\ln z + O(1), \quad k = 1, 2,$$

and carrying out the obvious transformations, we find

$$\Omega_1(z, z_0) = \frac{i(\bar{a}_1 - \bar{b}_1) - (\bar{a}_2 - \bar{b}_2)}{\pi\bar{\lambda}\,(z - z_0)} + \cdots,$$

$$\Omega_2(z, z_0) = \frac{i(\bar{a}_1 - \bar{b}_1) + (\bar{a}_2 - \bar{b}_2)}{\pi\bar{\lambda}\,(z - z_0)} + \cdots,$$

where the ellipses replace regular terms and terms with logarithmic singularity. In accordance with the requirements stated above, we obtain the conditions

$$i(a_1 - b_1) + (a_2 - b_2) = \pi\lambda, \quad i(a_1 - b_1) - (a_2 - b_2) = 0,$$

whence

$$2(a_1 - b_1) = -\pi i\lambda, \quad 2(a_2 - b_2) = \pi\lambda.$$

These conditions can be satisfied by putting

$$a_1 = -\frac{\pi}{2} i\lambda, \quad b_1 = 0, \quad a_2 = \frac{\pi}{2}\lambda, \quad b_2 = 0.$$

Then from formulas (4), (5), and (6) we obtain

$$\begin{aligned} \Omega_1(z, \zeta) &= \pi|\lambda| e^{-i\arg(z-\zeta)} H_1^{(1)}(2i|\lambda|r), \\ \Omega_2(z, \zeta) &= -\pi i\lambda H_0^{(1)}(2i|\lambda|r), \qquad r = |z - \zeta|. \end{aligned} \tag{7}$$

Here we have used the fact that $H_1^{(1)}(2i|\lambda|r)$ is real (see, for example, ([4]), p. 163). Since the functions (5) satisfy equation (2) with respect to the first argument z, from (6) we obtain the identities

$$\frac{\partial\Omega_1(z, \zeta)}{\partial\bar{z}} = \lambda\overline{\Omega_2(z, \zeta)}, \qquad \frac{\partial\Omega_2(z, \zeta)}{\partial\bar{z}} = \lambda\overline{\Omega_1(z, \zeta)}. \tag{8}$$

Finally, by direct verification we see that, with respect to the second argument ζ, the functions (7) satisfy the system

$$\frac{\partial\Omega_1(z, \zeta)}{\partial\bar{\zeta}} = -\bar{\lambda}\Omega_2(z, \zeta), \qquad \frac{\partial\Omega_2(z, \zeta)}{\partial\zeta} = -\lambda\Omega_1(z, \zeta). \tag{9}$$

Consequently, the functions (7) represent the kernels of the generalized Cauchy formula (see ([1]), Chapter 3).

4. Let Γ denote the aggregate of several rectifiable curves that do not intersect one another, and let G be the domain with boundary Γ, containing the point at infinity. Draw the circle Γ_R of radius R, enclosing Γ, with center at an arbitrarily chosen point $z \in G$. As shown in ([1]), p. 185, every solution $F(z)$ of equation (2) can be represented in the form of the generalized Cauchy formula:

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma+\Gamma_R} \Omega_1(z,\zeta) F(\zeta)\, d\zeta - \Omega_2(z,\zeta) \overline{F(\zeta)}\, d\bar{\zeta}, \tag{10}$$

where the contour $\Gamma + \Gamma_R$ is traversed in the positive direction. From (7) the following asymptotic representations of the kernels Ω_1 and Ω_2 follow:

$$\Omega_1(z,\zeta) = -\pi^{1/2} |\lambda|^{1/2} e^{-i \arg(z-\zeta)} r^{-1/2} e^{-2|\lambda| r} \left[1 + O\left(\frac{1}{r}\right)\right], \qquad r = |z-\zeta|,$$

$$\Omega_2(z,\zeta) = -i\pi^{1/2} \lambda |\lambda|^{-1/2} r^{-1/2} e^{-2|\lambda| r} \left[1 + O\left(\frac{1}{r}\right)\right]. \tag{11}$$

In order that the integral over Γ_R in formula (10) vanish, the solution $F(z)$ must have the following behavior at infinity:

$$F(z) = e^{2|\lambda| r} r^{-1/2} o(1). \tag{12}$$

Under these assumptions, from (10) we obtain

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \Omega_1(z,\zeta) F(\zeta)\, d\zeta - \Omega_2(z,\zeta) \overline{F(\zeta)}\, d\bar{\zeta}. \tag{13}$$

Let now $\mu(\zeta)$ be an arbitrary function summable on Γ. We construct a generalized Cauchy-type integral

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \Omega_1(z,\zeta) \mu(\zeta)\, d\zeta - \Omega_2(z,\zeta) \overline{\mu(\zeta)}\, d\bar{\zeta}. \tag{14}$$

By virtue of the identities (8), this will be a solution of equation (2), and from (11) it follows that the function (14) has the following behavior at infinity:

$$F(z) = r^{-1/2} e^{-2|\lambda| r} O(1), \qquad r = |z|. \tag{15}$$

In particular, a solution of equation (2) satisfying condition (12) in fact satisfies even condition (15).

As was shown in ([3]), p. 127, if a solution of equation (3) has the asymptotic behavior (15) with $O(1)$ replaced by $o(1)$, then $F(z) \equiv 0$. Here, as above, the condition $\lambda \neq 0$ is essential. If $\lambda = 0$, i.e., if analytic functions are considered, then an analogous condition of the form $F(z) = r^{-1/2} o(1)$, as is known, does not imply the identity $F(z) \equiv 0$. In this is manifested the peculiar character of the theory of **generalized** analytic functions, and this peculiarity is analogous to that noted in ([3]) when comparing equation (3) with the Laplace equation.

In ([3]), pp. 128-130, it is also shown that condition (15) is equivalent to the Sommerfeld condition, which, as applied to equation (3), has the form

$$\frac{\partial F}{\partial r} + 2|\lambda|F = e^{-2|\lambda|r} r^{-1/2} o(1). \tag{16}$$

Thus the following assertion has been proved:

Theorem. The functions $\Omega_1(z, \zeta)$, $\Omega_2(z, \zeta)$, constructed according to formulas (7), are the kernels of the generalized Cauchy formula for equation (2). Their asymptotic behavior at infinity is described by formulas (11); for $z = \zeta$ the function Ω_1 has a pole of the first order with residue 1, while Ω_2 has a logarithmic singularity. Every solution of equation (2) having at infinity the behavior (12) is representable in the infinite domain G in the form of the generalized Cauchy formula (13). This solution, as well as any generalized Cauchy-type integral (14), satisfies condition (15) or (16).

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Note: Figure translations are in progress. See original paper for figures.