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Abstract

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ALMOST GEODESIC MAPPINGS OF AFFINELY CONNECTED AND RIEMANNIAN SPACES

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At first the notion of almost geodesic lines of affinely connected spaces without torsion is introduced here, as curves closest to geodesic lines in the affine sense. Then mappings of affinely connected (and Riemannian) spaces are considered, under which almost geodesic or geodesic lines of one space pass into almost geodesic lines of another. The investigation is carried out locally in the class of analytic functions.

1. A curve C, defined by the equations $x^i = x^i(t)$ $(i = 1, 2, \dots, n)$, will be called **almost geodesic** of the affinely connected space A_n, if for it there exists a field of parallel planes $E_2\{\lambda^i, \mu^i\}$, each of which at the corresponding point passes through the tangent vector $\xi^i = dx^i/dt$. Thus an almost geodesic is characterized by the fact that along it the conditions are identically satisfied

$$\xi^i = p\lambda^i + q\mu^i; \quad \lambda^i{}_{,\alpha}\xi^\alpha = a_1\lambda^i + b_1\mu^i; \quad \mu^i{}_{,\alpha}\xi^\alpha = a_2\lambda^i + b_2\mu^i. \tag{1}$$

It follows from this that the differential equation of almost geodesics has the form

$$\xi^i_{2|} = a\xi^i + b\xi^i_{1|}, \tag{2}$$

where $\xi^i_{1|} = \xi^i{}_{,\alpha}\xi^\alpha$, $\xi^i_{2|} = \xi^i_{1|,\alpha}\xi^\alpha$, the comma denotes the sign of covariant differentiation. Here a and b may be regarded as arbitrary functions of $x^1, x^2, \dots, x^n, t$. For given a and b, system (2) has a unique solution for any initial values $x^i|_{t=t_0}$, $\xi^i|_{t=t_0}$, $\xi^i_{1|}|_{t=t_0}$. When A_n is flat, its almost geodesics are curves lying in two-dimensional planes (^{1}).

2. Considering a mapping $\bar{A}_n$ onto A_n, under which every almost geodesic of one passes into an almost geodesic of the other, it is not hard to find that it must necessarily be geodesic. Obviously, the converse is also true.

3. We therefore turn to a mapping $\bar{A}_n$ onto A_n, under which every geodesic of the first passes into an almost geodesic of the second, calling it **almost geodesic**. Obviously, in equation (2) of the almost geodesic of A_n, corresponding to a geodesic of $\bar{A}_n$, a and b depend on the direction of the latter. Assuming this dependence to be analytic (and then it will necessarily be rational), we find that an almost geodesic mapping is characterized by the conditions

$$P^h_{(ij,k)} + 2P^h_{\alpha(i}P^\alpha_{jk)} = a_{(ij}\delta^h_{k)} + b_{(i}P^h_{jk)}. \tag{3}$$

Here $P^h_{ij} = \Gamma^h_{ij} - \bar{\Gamma}^h_{ij}$, Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$ are the connection coefficients of A_n and $\bar{A}_n$ in a common coordinate system under the mapping, a_{ij} is a symmetric tensor, b_i is a vector, and parentheses denote cyclic summation. When A_n is flat and referred to affine coordinates, (3) gives the fundamental equations of the theory of $n-2$ projective spaces (^{2}). These equations are satisfied by $\bar{\Gamma}^k_{ij} = fC^k_{ij}$, where f is an arbitrary function, and C^k_{ij} are constants connected by the conditions $C^h_{\alpha(i}C^\alpha_{jk)} = 0$, i.e., they define a commutative algebra satisfying the Jacobi identity. The spaces (^{3}) belong here.

4. An almost geodesic mapping $\bar{A}_n$ onto A_n will be called **linear** if to each geodesic of $\bar{A}_n$ there corresponds an almost geodesic of A_n, for which the field of parallel planes E_2 at each point is linearly

depends on the direction of the tangent. In this case one may assume that $\mu^i = \bar{\mu}^i + \mu^i_\alpha\xi^\alpha$, $\lambda^i = \xi^i$, where $\bar{\mu}^i$ and μ^i_j depend only on the point. From conditions (1), under analytic dependence of the coefficients on ξ^i, it follows that there exist only two types of linear almost geodesic mappings. For one of them $\mu^i = \mu^i_\alpha\xi^\alpha$,

$$P^k_{ij} = \varphi_i\delta^k_j + \varphi_j\delta^k_i + \psi_i\mu^k_j + \psi_j\mu^k_i,$$

$$\mu^i_{j,k} + \mu^i_{k,j} + 2(\psi_j\mu^i_\alpha\mu^\alpha_k + \psi_k\mu^i_\alpha\mu^\alpha_j) = \rho_j\delta^i_k + \rho_k\delta^i_j + \sigma_j\mu^i_k + \sigma_k\mu^i_j, \tag{4}$$

and for the other $\mu^i = \bar{\mu}^i$,

$$P^k_{ij} = \varphi_i\delta^k_j + \varphi_j\delta^k_i + \psi_{ij}\mu^k, \qquad \mu^i{}_{,j} = \rho\delta^i_j + \sigma_j\mu^i. \tag{5}$$

Here $\varphi_i, \psi_j, \sigma_k, \rho_j, \mu^i$ are vectors, μ^i_j is an affinor, and ψ_{ij} is a symmetric tensor. Equations (4) also give the additional relations

$$\psi_{j,k} + \psi_{k,j} = \nu_j\psi_k + \nu_k\psi_j \tag{6}$$

for $\mu^i_j \neq \tau\delta^i_j + \xi^i\theta_j$ for the first type, and

$$\psi_{(jkl)} = 0, \qquad \psi_{jkl} = \psi_{jk,l} - \nu_l\psi_{jk} \tag{7}$$

for $\mu^i \neq 0$ for the second.

If, in item 3, the requirement of analyticity is somewhat weakened, then conditions (6) and (7) disappear.

5. Finally, let us consider linear almost geodesic mappings of type II, distinct from geodesic mappings, for Riemannian spaces $\bar{V}_n$ and V_n (with metric tensors $\bar{g}_{ij}$ and g_{ij}). In this case, in V_n there must exist a vector $\mu^i \neq 0$ satisfying conditions (5). If it is nonisotropic, by normalization one can pass to the gradient vector $\tilde{\mu}_i = \partial\mu/\partial x^i$ and, in a special coordinate system, reduce the metric form of the space to the form $ds^2 = e\,dx^{1\,2} + F\,d\tilde{s}^2$, where $e = \pm 1$, F is an arbitrary function of $x^1, x^2, \dots, x^n$, and $d\tilde{s}^2$ is an arbitrary metric of $\tilde{V}_{n-1}$ in the manifold $x^2, x^3, \dots, x^n$.

These conditions are also sufficient, since a conformal mapping of V_n by means of any function of $\tilde{\mu}$ turns out to be a linear almost geodesic mapping of type II (and conditions (7) are satisfied automatically). If μ^i in (5) is isotropic, then $\rho = 0$, and it is absolutely parallel. When σ_j is a gradient, analogously to the preceding case we arrive at the gradient vector $\tilde{\mu}_i$, at a conformal mapping that is a linear almost geodesic mapping of type II, and at the well-known canonical form of the metric form of the space. When one of the spaces in a linear almost geodesic correspondence of type II is flat, the spaces (4) correspond to the solution found.

If, similarly to the preceding, one considers curves of affinely connected spaces for which the minimal dimension of the field of parallel planes E_k passing through the tangent vector is $k > 2$ $(< n)$, one obtains an analogous generalization of the theory of $n-k$ projective spaces.

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