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Abstract

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THREE-DIMENSIONAL HOMOGENEOUS EXPANSION–COMPRESSION OF A RAREFIED GAS WITH POWER-LAW INTERACTION FUNCTIONS

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All quantities in the paper are assumed to be dimensionless.

In paper (^{1}) the motions of three-dimensional homogeneous expansion–compression in an unbounded space of a monatomic gas or of mixtures of monatomic gases were considered, when the functions f_i of the distribution of the peculiar velocities of the molecules are the same for all points of space and depend only on time. The Boltzmann equations for the functions $f_i = f_i(t, c_i)$ of the distribution of the velocity vectors of molecules of the i-th species then have the form

$$\frac{\partial f_i}{\partial t} - \frac{1}{t}\left(u_i \frac{\partial f_i}{\partial u_i} + v_i \frac{\partial f_i}{\partial v_i} + w_i \frac{\partial f_i}{\partial w_i}\right) = I_i(t, c_i), \tag{1}$$

where u_i, v_i, w_i are the components of the vectors c_i; t is time. The collision integrals I_i, as usual, have the form (^{2})

$$I_i(t, c_i) = \sum_j \iiint [f_i(t, c_i') f_j(t, c_j') - f_i(t, c_i) f_j(t, c_j)] q_{ij} b \, db \, d\varepsilon \, dc_j. \tag{2}$$

Here $g_{ij} = |c_i - c_j|$; b is the dimensionless impact distance; $dc_j = du_{jdv_{jdw}}j$; ε is the angle of rotation in the planes normal to the vector $c_i - c_j$; c_i', c_j' are the velocity vectors after collision of particles that before the collision had velocity vectors respectively c_i, c_j.

Let us consider a model of point centers of potential repulsion or attraction. For it, on the basis, for example, of the book (^{2}) (Chap. III, § 4), we have the relations

$$c_i' = \varphi_{ij}(c_i, c_j, \chi_{ij}, e) = \frac{m_i c_i + m_j c_j}{m_0} + \frac{m_j}{m_0}[(c_i - c_j)\cos\chi_{ij} - e g_{ij} \sin\chi_{ij}]; \tag{3}$$

$$c'_j = \psi_{ij}(c_i, c_j, \chi_{ij}, e) = \frac{m_i c_i + m_j c_j}{m_0} - \frac{m_i}{m_0}[(c_i - c_j)\cos\chi_{ij} - e g_{ij}\sin\chi_{ij}]. \tag{4}$$

Here m_i and m_j are the masses of the colliding particles; $m_0 = m_i + m_j$; $\chi_{ij} = \chi_{ij}(b, g_{ij})$ is the angle of rotation of the vector $c_i - c_j$ as a result of the collision; e is a unit vector orthogonal to the vector $c_i - c_j$. The magnitude of the increment of this vector is equal to the quantity $d\varepsilon$ in relation (2) and has the same meaning.

Suppose now that between particles i and particles j there acts a force $P_{ij} = \varkappa_{ij}/r^{\nu}$, where r is the distance between the particles, the quantity ν is constant in all cases, and the constants $\varkappa_{ij}$ either all have the same sign, or their sign depends on the combination of indices i and j. In the case under consideration the angle χ_{ij} is determined by the expression (see (^{2}), Chap. X, § 3):

$$\chi_{ij} = \chi_{ij}(b, g_{ij}) = \pi - 2\int_0^{v_{00}} \left\{ 1 - v^2 - \frac{2}{\nu - 1}\left(\frac{v}{v_0}\right)^{\nu-1} \operatorname{sign}\varkappa_{ij} \right\}^{-1/2} dv, \tag{5}$$

where v_{00} is the root of the equation

$$1 - v^2 - \frac{2}{\nu - 1}\left(\frac{v}{v_0}\right)^{\nu-1} \operatorname{sign}\varkappa_{ij} = 0,$$

and the quantity v_0, on which alone χ_{ij} depends, is determined by the relation

$$v_0 = b\left(\frac{m_i m_j g_{ij}^2}{m_0 |\varkappa_{ij}|}\right)^{1/(\nu-1)}. \tag{6}$$

By virtue of the linearity of relations (3), (4), we have the similarity property

$$\begin{aligned} \varphi_{ij}(\lambda c_i, \lambda c_j, \chi, e) &= \lambda \varphi_{ij}(c_i, c_j, \chi, e), \\ \psi_{ij}(\lambda c_i, \lambda c_j, \chi, e) &= \lambda \psi_{ij}(c_i, c_j, \chi, e), \end{aligned} \tag{7}$$

where λ is an arbitrary positive quantity. Relations (5), (6) give one more similarity property

$$\chi_{ij}(b, g_{ij}/\lambda) = \chi_{ij}(\lambda^{-2(\nu-1)} b, g_{ij}). \tag{8}$$

Now, in the same way as we did for the case of hard elastic spheres in [1], let us try to associate with each homogeneous state of a gas of the type under consideration (when the macroscopic velocities are equal to zero) some motion of expansion–compression of the same gas. In any homogeneous state the functions $f_i = f_i^0(t, c_i)$ of the velocity distribution c_i satisfy the Boltzmann equations

$$\frac{\partial f_i^0(t,c_i)}{dt} = \sum_j \iiint \left[f_i^0(t,c_i') f_j^0(t,c_j') - f_i^0(t,c_i) f_j^0(t,c_j) \right] g_{ij} b \, db \, d\varepsilon \, dc_j = I_i^0(t,c_i). \tag{9}$$

We shall seek a solution of equations (1) in the form

$$f_i = f_i^0(z, A_i), \qquad A_i = |t| c_i, \qquad z = z(t), \tag{10}$$

where $z(t)$ is to be determined. Then equalities (2) take the form

$$I_i(t,c_i) = \sum_j \iiint \{ f_i^0(z,|t|c_i') f_i^0(z,|t|c_j') -$$

$$- f_i^0(z,|t|c_i) f_i^0(z,|t|c_j) \} \, g_{ij} \, db \, d\varepsilon \, dc_j, \tag{11}$$

where c_i', c_j' are determined by relations (3), (4), in which for χ_{ij} relations (5), (6) hold, and the similarity properties (7), (8) take place. Our considerations are valid only for $\nu > 2$, for at $\nu \leqslant 2$ the integrals I_i diverge (see [2], Ch. 10, § 3, item 3). Setting $\lambda = |t|$ in equalities (7), (8), we obtain:

$$|t| \, c_i' = \varphi_{ij}[A_i, A_j, \chi_{ij}(b_1, G_{ij}), e], \tag{12}$$

$$|t| \, c_j' = \psi_{ij}[A_i, A_j, \chi_{ij}(b_1, G_{ij}), e], \tag{13}$$

where $G_{ij} = |t| g_{ij} = |A_i - A_j|$, $b_1 = |t|^{-2/(\nu-1)} b$.

Introducing in the right-hand side of (11) new variables of integration according to the formulas $b_1 = |t|^{-2/(\nu-1)} b$, $dA_j = |t|^3 dc_j$, we obtain

$$I_i(t,c_i) = |t|^{4\frac{2-\nu}{\nu-1}} I_i^0[z(t), A_i]. \tag{14}$$

The left-hand side of equation (1) takes the form:

$$\frac{\partial f_i}{\partial t} - \frac{1}{t}\left(u_i \frac{\partial f_i}{\partial u_i} + v_i \frac{\partial f_i}{\partial v_i} + w_i \frac{\partial f_i}{\partial w_i} \right) = \frac{\partial f_i^0(z, A_i)}{\partial z} \frac{dz(t)}{dt}. \tag{15}$$

Taking relations (14) and (15) into account, we reduce equation (1) to the form

$$\frac{dz(t)}{dt}\frac{\partial f_i^0(z,A_i)}{\partial z}=|t|^{4\frac{2-\nu}{\nu-1}}I_i^0(z,A_i). \tag{16}$$

However, by virtue of equations (8), (3), (4), (5), (6), (9) we have $\partial f_i(t,c_i)/\partial t = I_i^0(t,c_i)$, where, obviously, an arbitrary positive scalar quantity may be substituted for t, and an arbitrary vector for c_i. Therefore we have $\partial f_i^0(z,A_i)/\partial z = I_i^0(z,A_i)$, and equations (16) will be satisfied if we set

$$\frac{dz(t)}{dt}=|t|^{4\frac{2-\nu}{\nu-1}}. \tag{17}$$

For the case of expansion $t>0$, we have:

$$z(t)=\beta+\frac{\nu-1}{7-3\nu}t^{\frac{7-3\nu}{\nu-1}},\qquad \beta=\text{const}. \tag{18}$$

Thus, in the case of spreading we have

$$f_i(t,c_i)=f_i^0\left[\left(\beta+\frac{\nu-1}{7-3\nu}t^{\frac{7-3\nu}{\nu-1}}\right),tc_i\right]. \tag{19}$$

The solution in the case of expansion, when at $t=1$ (this is the general case, in view of the arbitrariness of the time scale) $f_i=\Phi_i(c_i)$, is obtained in the form

$$f_i=f_i(t,c_i)=f_i^0\left[\left(\frac{\nu-1}{3\nu-7}+\frac{\nu-1}{7-3\nu}t^{\frac{7-3\nu}{\nu-1}}\right),tc_i\right], \tag{20}$$

where $f_i^0(t,c_i)$ is the solution for the homogeneous state with initial data $f_i^0(0,c_i)=\Phi_i(c_i)$. The expressions given are not suitable for $\nu=7/3$. Here we have $z(t)=\ln(t/\beta)$, where $\beta=\text{const}$. The solution of the problem posed above has the form

$$f_i=f_i(t,c_i)=f_i^0(\ln t,tc_i). \tag{21}$$

The quantity $\dfrac{\nu-1}{3\nu-7}\left(1-t^{\frac{7-3\nu}{\nu-1}}\right)>0$ for $t>1$ and for any ν increases as t increases. For $\nu>7/3$, as $t\to\infty$ it tends to a definite limit, namely the value $(\nu-1)/(3\nu-7)$. Equality (20) gives

$$f_i(\infty,c_i)=f_i^0\left(\frac{\nu-1}{3\nu-7},tc_i\right). \tag{22}$$

Thus, under expansion, in infinite time a distribution is attained that corresponds to the distribution attained for the homogeneous state already at $t=(\nu-1)/(3\nu-7)$. For $\nu\leqslant 7/3$ we have

$$f_i(\infty, c_i) = f_i^0(\infty, tc_i). \tag{23}$$

The distribution $f_i^0(t, c_i)$ as $t \to \infty$ tends to the Maxwell distribution (²). Therefore, for $\nu < 7/3$, under expansion the velocity distribution also tends to the Maxwellian.

For the case of compression $t < 0$, $dz(t)/dt = -dz(t)/d|t|$, integration of equation (17) gives

$$z(t) = \beta - \frac{\nu - 1}{7 - 3\nu}|t|^{\frac{7-3\nu}{\nu-1}}. \tag{24}$$

Analogously to the preceding, we obtain that the solution for the case of compression, coinciding at $t = -1$ with the homogeneous state, has the form

$$f_i(t, c_i) = f_i^0\left[\left(\frac{\nu - 1}{7 - 3\nu} - \frac{\nu - 1}{7 - 3\nu}|t|^{\frac{7-3\nu}{\nu-1}}\right), |t|c_i\right]. \tag{25}$$

For $\nu \geqslant 7/3$, the velocity distribution tends to the Maxwellian as $t \to -0$. For $\nu < 7/3$, as $t \to -0$ it tends to the distribution

$$f_i = f_i^0\left(\frac{\nu - 1}{7 - 3\nu}, |t|c_i\right)$$

and thus does not attain the Maxwell distribution, despite the fact that the particle density per unit volume increases as the quantity $1/|t|^3$. Apparently this is explained by the fact that the growth of density in the latter case does not compensate for the boundedness of the time interval $-1 < t < 0$. Formally putting $\nu \to \infty$, we obtain the results of article (¹), which pertain to spherical elastic molecules.

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2. S. Chapman, T. Cowling, *The Mathematical Theory of Non-Uniform Gases*, IL, 1960.

Note: Figure translations are in progress. See original paper for figures.

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