Mathematics

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Abstract

Full Text

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ON THE THEORY OF TESTS FOR TWO NORMAL SAMPLES

The present note generalizes or strengthens the results of several preceding notes ($^{1-3}$) concerning tests for comparing two normal samples. Let $x_1, \dots, x_{n_1} \in N(a_1, \sigma_1)$; $y_1, \dots, y_{n_2} \in N(a_2, \sigma_2)$ be two normal repeated samples with different variances σ_1^2 and σ_2^2. The hypothesis of equality of the means $H_0 : a_1 = a_2$ is tested by means of similar, with respect to the parameters, tests randomized in the final decision, i.e. producing a function of the samples $\Phi(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \equiv \Phi(x, y)$ under the condition $0 \leq \Phi(x, y) \leq 1$—the probability of rejecting H_0 after the observations. If $\Phi(x, y)$ takes only the values 0 and 1 with probability 1, we shall call the test nonrandomized.

The problem of describing all randomized similar measurable tests is very difficult and has not yet been solved. Much work has been devoted to this question; individual constructions are known—nonrandomized Bartlett-Romanovsky-Scheffé tests (^{1,5}), randomized Welch tests (^{6}), approximately similar tests depending only on sufficient statistics—R. Welch (^{7}), A. Wald (^{8}).

The sufficient statistics of the problem are the well-known quantities $\bar{x}, \bar{y}, s_1^2, s_2^2$. If $\Phi = \Phi(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})$ is some similar test of size α,

$$\mathbf{E}(\Phi(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \mid H_0) = \alpha, \tag{1}$$

then we may consider the conditional mathematical expectation:

$$\widetilde{\Phi}(\bar{x}, \bar{y}, s_1, s_2) = \mathbf{E}(\Phi(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \mid \bar{x}, \bar{y}, s_1, s_2, H_0), \tag{2}$$

and, obviously,

$$\mathbf{E}(\widetilde{\Phi}(\bar{x}, \bar{y}, s_1, s_2) \mid H_0) = \alpha \tag{3}$$

for all values of the parameters, so that the new randomized test will also be similar. Thus we may try to classify similar tests $\Phi(x, y)$ by considering their "projections" (2) onto the σ-algebra of sufficient statistics (when "projecting" onto other σ-algebras we would not obtain tests, since unknown parameters would appear). If, for example, we take the simplest similar test of Bartlett type, assuming $n_1 = n_2$:

$$\Phi(x, y) = 0, \quad \text{if } |\bar{x} - \bar{y}| \left(\sum_{i=1}^{n_1} \left[(x_i - \bar{x}) - (y_i - \bar{y}) \right]^2 \right)^{1/2} \leq \varepsilon_0, \qquad (4)$$

$$\Phi(x, y) = 1 \quad \text{in the case of the opposite inequality,}$$

and consider the "projection" (2) onto the σ-algebra of sufficient statistics, which we denote by $\widetilde{\Phi}(\bar{x}, \bar{y}, s_1, s_2)$, then the test $\widetilde{\Phi}$ will be a randomized and similar test in the domain of sufficient statistics and, apparently, not a test of Welch type.*

* Such a "projection" is easy to calculate, using the fact that the sample correlation coefficient is stochastically independent of the sufficient statistics.

We see that test (4) is constructed by means of the linear form $\bar{x} - \bar{y}$, "standardized" by means of the quadratic form

$$Q = \sum_{i=1}^{n_i} [(x_i - \bar{x}) - (y_i - \bar{y})]^2.$$

It turns out, however, that there is no similar test in which such standardization would be carried out by means of sufficient statistics. More precisely, the following theorem holds:

Theorem 1. *There does not exist a randomized nontrivial similar test* $\Phi(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})$ *which, with probability 1, would accept the null hypothesis* H_0 *if*

$$|\bar{x} - \bar{y}| \, (s_1^2 + s_2^2)^{-1/2} \leqslant \varepsilon_0 \qquad (5)$$

for sufficiently small ε_0.

In note ([3]) the nonexistence of an exact (similar) A. Wald test ([8]) was proved under rather stringent assumptions concerning the test function Φ in Wald' s test with critical region:

$$|\bar{x} - \bar{y}| \, (s_1^2 + s_2^2)^{-1/2} \geqslant \Phi(s_1 \mid s_2) \qquad (6)$$

(the samples are assumed to be of equal size $n_1 = n_2 = n$).

We can formulate the same theorem under considerably weaker assumptions.

Theorem 2. *There does not exist a similar nonrandomized A. Wald test (6) with equal sample sizes, where the function* $\Phi = \Phi(\eta)$ *has an everywhere finite* *first derivative* $\Phi'(\eta)$ *on the interval* $(0, 1)$ *and satisfies the Lipschitz condition* $\text{Lip}^{(1)}$ *on the segment* $[0, \eta_0]$ *for sufficiently large* η_0.

The proofs of these two theorems are carried out by the method of analytic continuation with respect to a parameter, as outlined in notes ([1]-[3]).

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Note: Figure translations are in progress. See original paper for figures.