On the Theory of Ehresmann Jets

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Abstract

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Playing an important role in the foundations of differential geometry, the theory of Ehresmann jets is usually constructed as a topological theory ($^{1-3}$). In the present note the possibility is shown of constructing this theory in a more general form, without assuming the base sets to be topological spaces.

Let (M, ω) be an arbitrary ordered set. An equivalence relation ε that is the equivalence closure (^{4}) of the order relation ω is called the **connectedness relation**, and the classes of the partition of the set M determined by it are called the **connected components of the ordered set** (M, ω). The binary relations $\omega \circ \omega^{-1}$ and $\omega^{-1} \circ \omega$ are called, respectively, the **relation of minorization** and the **relation of majorization**. It can be shown that the connectedness relation coincides with the transitive closure of the relation of minorization or of the relation of majorization. If $\omega \circ \omega^{-1} = M \times M$, then, as is known (^{5}), the ordered set (M, ω) is called **filtered to the left**.

We shall call an ordered set (M, ω) **partially filtered to the left** if each of its connected components is an ordered set filtered to the left. It can be shown (^{6}) that the fact that an ordered set (M, ω) is partially filtered to the left is equivalent to the fact that its relation of minorization is an equivalence relation, which, evidently, will coincide with the connectedness relation, or to the fact that its relation of majorization is contained in the relation of minorization $\omega^{-1} \circ \omega \subset \omega \circ \omega^{-1}$.

A **centered subset** of a set A is any pair $(a, \mathfrak{a})$, where a is an element of the subset $\mathfrak{a}$. The set of centered subsets $E_{\mathfrak{A}}$, defined by the formula

$$(a, \mathfrak{a}) \in E_{\mathfrak{A}} \leftrightarrow a \in \mathfrak{a} \wedge \mathfrak{a} \in \mathfrak{A}, \tag{1}$$

where $\mathfrak{A} \subset \mathfrak{P}(A)$ is a given set of subsets, will be a binary relation between the elements of the set A and its subsets, which is called the **partial membership relation associated with the set of subsets** $\mathfrak{A}$. Evidently, the section $E_{\mathfrak{A}}\langle a \rangle$ of the binary relation $E_{\mathfrak{A}}$ through the element a is the set of all subsets in $\mathfrak{A}$ to which the element a belongs.

A nonempty set of subsets $\mathfrak{A}$ is called **localizable** if every nonempty section $E_{\mathfrak{A}}\langle a \rangle$, where $a \in \bigcup \mathfrak{A}$, considered as a set ordered by the inclusion relation, is partially filtered to the left.

Theorem 1. *The localizability of a set of subsets $\mathfrak{A} \subset \mathfrak{P}(A)$ is equivalent to the condition that the intersection of any majorized pair of subsets from $\mathfrak{A}$ coincides with the union of some set of subsets from $\mathfrak{A}$.*

Proof. Denoting by ω_a the order relation in $E_{\mathfrak{A}}\langle a\rangle$, where $a \in \bigcup \mathfrak{A}$, determined by the inclusion relation, we can

the localizability condition for $\mathfrak{A}$ can be expressed by the formula

$$\bigwedge_{a \in \bigcup \mathfrak{A}} \left(\omega_a^{-1} \circ \omega_a \subset \omega_a \circ \omega_a^{-1}\right). \tag{2}$$

For any $a_1, a_2 \in \mathfrak{A}$ we have

$$(a_1, a_2) \in \omega_a^{-1} \circ \omega_a \leftrightarrow a \in a_1 \cap a_2 \wedge \bigvee_{a \in \mathfrak{A}} (a_1 \subset a \wedge a_2 \subset a),$$

$$(a_1, a_2) \in \omega_a \circ \omega_a^{-1} \leftrightarrow \bigvee_{a \in \mathfrak{A}} (a \in a \wedge a \subset a_1 \cap a_2), \tag{3}$$

whence

$$\bigwedge_{a \in \bigcup \mathfrak{A}} (\omega_a^{-1} \circ \omega_a \subset \omega_a \circ \omega_a^{-1}) \leftrightarrow \bigwedge_{a_1, a_2 \in \mathfrak{A}} \bigwedge_{a \in \bigcup \mathfrak{A}} \left(a \in a_1 \cap a_2 \wedge \bigvee_{a \in \mathfrak{A}} (a_1 \subset a \wedge a_2 \subset a) \right) \rightarrow$$

$$\rightarrow \bigvee_{a \in \mathfrak{A}} (a \in a \wedge a \subset a_1 \cap a_2) \leftrightarrow \bigwedge_{a_1, a_2 \in \mathfrak{A}} \left(\bigvee_{a \in \mathfrak{A}} (a_1 \subset a \wedge a_2 \subset a) \rightarrow \bigvee_{\mathfrak{A} \subset \mathfrak{A}} \left(a_1 \cap a_2 = \bigcup \mathfrak{A}\right) \right),$$

which gives the proof of the theorem.

We shall regard the Cartesian product $A \times \mathfrak{P}(A)$ as a set ordered by means of the direct product of the identity binary relation between the elements of A and the inclusion relation in $\mathfrak{P}(A)$. Using the infix notation for this order relation in $A \times \mathfrak{P}(A)$ by the sign $\prec$, we thus have

$$(a_1, \mathfrak{a}_1) \prec (a_2, \mathfrak{a}_2) \leftrightarrow a_1 = a_2 \wedge \mathfrak{a}_1 \subset \mathfrak{a}_2. \tag{4}$$

Next we shall regard $E_{\mathfrak{A}}$, which is a subset of the ordered set $A \times \mathfrak{P}(A)$, also as an ordered set.

Theorem 2. *The localizability of a set of subsets $\mathfrak{A} \subset \mathfrak{P}(A)$ is equivalent to the fact that $E_{\mathfrak{A}}$ is a partially left-filtering ordered set.*

Proof. Denoting by ω the order relation in $E_{\mathfrak{A}}$, we obtain, according to (3) and (4), for any $(a_1, \mathfrak{a}_1), (a_2, \mathfrak{a}_2) \in E_{\mathfrak{A}}$,

$$((a_1, \mathfrak{a}_1), (a_2, \mathfrak{a}_2)) \in \omega^{-1} \circ \omega \leftrightarrow a_1 = a_2 \wedge (\mathfrak{a}_1, \mathfrak{a}_2) \in \omega_{a_1}^{-1} \circ \omega_{a_1},$$

$$((a_1, \mathfrak{a}_1), (a_2, \mathfrak{a}_2)) \in \omega \circ \omega^{-1} \leftrightarrow a_1 = a_2 \wedge (\mathfrak{a}_1, \mathfrak{a}_2) \in \omega_{a_1} \circ \omega_{a_1}^{-1},$$

whence

$$\omega^{-1} \circ \omega \subset \omega \circ \omega^{-1} \leftrightarrow \bigwedge_{a \in \bigcup \mathfrak{A}} \omega_a^{-1} \circ \omega_a \subset \omega_a \circ \omega_a^{-1}.$$

If $\mathfrak{A}$ is a localizable set of subsets, then the set of connected components $E_{\mathfrak{A}}$ will be called the **localizer** of $\mathfrak{A}$ and denoted by $L_{\mathfrak{A}}$.

Suppose that a partial n-operation o is given on the set A. Extending it to the set of subsets, we obtain in $\mathfrak{P}(A)$ an n-operation o, called the **global operation associated** with o. Let the localizable set of subsets $\mathfrak{A}$ be stable with respect to the global operation o; then $E_{\mathfrak{A}}$ will be a stable subset with respect to the direct product $o \Box o$. Denote by o_E the partial n-operation in $E_{\mathfrak{A}}$ induced by the partial n-operation $o \Box o$.

It can be shown that the connectedness relation

$$\varepsilon = \omega \circ \omega^{-1}$$

will be a stable equivalence relation with respect to o_E, i.e., a stable sub-

by a set relative to the direct square $o_E \Box o_E$. Hence it follows that in the localizer $L_{\mathfrak{A}}$, stable with respect to the global operation o of the localizable set of subsets $\mathfrak{A}$, there is defined a partial n-operation o_L, which is a partial factor-operation of the partial n-operation o_E under the equivalence ε, stable with respect to its relation. We shall call the partial n-operation o_L in the localizer $L_{\mathfrak{A}}$ **locally induced by means of the partial n-operation o in** A. Take, as the set A in the preceding arguments, the Cartesian product $B \times C$, and as the set of subsets $\mathfrak{A}$ the set $\Phi \subset \mathfrak{F}(B, C)$ of partial mappings. Denote by Δ_B the identity $(n+1)$-relation among the elements of the set B. Obviously, Δ_B will be a partial n-operation in B. Suppose further that in the set C an arbitrary partial n-operation o is given. Introduce in the Cartesian product $B \times C$ the partial n-operation $\Delta_B \Box o$, which is the direct product of Δ_B and o. If now Φ is localizable and stable with respect to the global operation

$$\Delta_B \Box o,$$

then the partial n-operation in the localizer L_Φ, locally induced by means of $\Delta_B\Box o$, will be called **locally induced by means of the partial n-operation o in C.**

We shall call a **semicentered partial mapping** of the set B into the set C any pair (b, φ), where $b \in pr_1\varphi$ and $\varphi \in \mathfrak{F}(B, C)$. Obviously, to every semicentered partial mapping (b, φ) there will correspond bijectively the centered partial mapping $((b, \varphi(b)), \varphi)$. If $\Phi \subset \mathfrak{F}(B, C)$ is a localizable set of partial mappings, then, by virtue of the one-to-one correspondence between semicentered and centered partial mappings, to the localizer L_Φ there corresponds the set $\mathfrak{F}_\Phi$ of subsets of semicentered partial mappings, called jets. Moreover, if in C a partial n-operation o is given such that Φ is stable with respect to $\Delta_B\Box o$, then to the locally induced partial n-operation o_L there will correspond an isomorphic partial n-operation in the jet set $\mathfrak{F}_\Phi$, which we shall likewise call **locally induced by means of o.** Thus one obtains a direct generalization of Ehresmann' s theory of jets and of the theory of bundles based on it, i.e. of jet sets with locally induced partial operations, in which only the special case of localizable sets of partial transformations is considered, namely, the set of partial transformations $\Phi \subset \mathfrak{F}(B, C)$, where B is a topological space and C is some arbitrary set satisfying the condition that the partial mappings from Φ are defined on open subsets and that the restriction of each partial transformation from Φ to any open subset also belongs to Φ (^{3}).

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REFERENCES

[1] Ch. Ehresmann, *Géométrie differentielle*, Coll. Intern. Centre Nat. Rech. Sci., Strasbourg, 1953, Paris, 1953, p. 97.
[2] P. Dedecker, *Bull. Soc. Math. Belg.*, **6**, 97 (1953).
[3] P. Dedecker, *Mém. Soc. Roy. Sci. Liège*, **19**, No. 1 (1957).
[4] J. Riguet, *Bull. Soc. Math. France*, **76**, 114 (1948).
[5] N. Bourbaki, *Set Theory* (summary of results), in the book by N. Bourbaki, *General Topology*, Moscow, 1958.
[6] V. V. Vagner, *Mat. sborn.*, **52** (94), No. 1, 597 (1960).

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