ON THE STABILITY OF PERIODIC SOLUTIONS BIFURCATING FROM AN EQUILIBRIUM STATE

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Abstract

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MATHEMATICS

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ON THE STABILITY OF PERIODIC SOLUTIONS BIFURCATING FROM AN EQUILIBRIUM STATE

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1. Consider the system of ordinary differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n; \lambda) \qquad (i = 1, \dots, n), \tag{1}$$

whose right-hand sides depend on the scalar parameter λ. It will be convenient for us to write this system in vector form

$$\frac{dx}{dt} = f(t, x; \lambda). \tag{2}$$

In what follows it is assumed that the right-hand sides of the system are sufficiently smooth for small $x_1, \dots, x_n$. It is also assumed that

$$f(t, 0; \lambda) \equiv 0, \qquad f(t + \omega, x; \lambda) \equiv f(t, x; \lambda). \tag{3}$$

The first of conditions (3) means that system (1) has the zero solution (equilibrium state) for all values of λ. We shall be interested in the question of for which values of the parameter λ, in a neighborhood of the equilibrium state, nonzero periodic solutions appear, and in the question of the stability or instability of these periodic solutions.

Let $\varphi(t, x_0; \lambda)$ denote the solution of system (2) satisfying the initial condition

$$\varphi(0, x_0; \lambda) \equiv x_0, \tag{4}$$

and put

$$A(x; \lambda) = \varphi(\omega, x; \lambda). \tag{5}$$

The operator $A(x; \lambda)$ (the Poincaré–Andronov point-transformation operator) is defined in a neighborhood of the zero point θ for all the parameter values

λ under consideration. As is known and as is easy to see, the fixed points of the operator $A(x;\lambda)$ are the initial conditions of the ω-periodic solutions of the system.

In a report at the IV All-Union Mathematical Congress I noted that the periodic solution corresponding to a fixed point x^* of the operator $A(x;\lambda)$ is asymptotically stable if x^* lies in some cone K on which the operator $A(x;\lambda)$ is positive and u_0-concave (see ([1])). This idea underlies the nonlocal theorems on the existence of asymptotically stable positive solutions presented in ([2]).

It is not difficult to see that convexity of the operator $A(x;\lambda)$ leads to instability of the corresponding periodic solution.

It turns out that, under fairly general assumptions, the operator (5) leaves invariant, in a neighborhood of zero, certain cones. Moreover, on these cones the operator (5) has the property of convexity or concavity depending on the sign of a certain integral, whose computation does not require knowledge of the periodic solutions. Thus, from the sign of a certain integral one can judge the asymptotic stability or instability of the periodic solutions bifurcating, as the parameter varies, from the equilibrium state.

2. By $B(t;\lambda)$ denote the ω-periodic matrix with elements

$$b_{ij}(t;\lambda) = \frac{\partial}{\partial x_j} f_i(t,0,\dots,0) \qquad (i,j=1,\dots,n). \tag{6}$$

The solution of the linear system

$$\frac{dx}{dt} = B(t;\lambda)x, \tag{7}$$

satisfying the initial condition $x(0)=x_0$, will be denoted by

$$x(t) = U(t;\lambda)x_0. \tag{8}$$

If $B(t;\lambda)$ does not depend on t, then $U(t;\lambda) = \exp\{tB(\lambda)\}$. The matrix $U(\omega;\lambda)$ is called the monodromy matrix, and its eigenvalues μ are the multipliers of system (7) (for these concepts see, for example, ([3])).

We shall use this occasion to indicate one new method for the approximate computation of multipliers, not connected with integration of system (7).

Introduce the operator

$$\Pi_\mu x(t) = \frac{1}{\mu\omega}\int_0^\omega [I + sB(s)]x(s)\,ds + \int_0^t B(s)x(s)\,ds. \tag{9}$$

It is not difficult to see that the solutions of the equation $x(t) = \Pi_\mu x(t)$ are solutions of system (7) whose initial conditions are eigenvectors of the monodromy matrix corresponding to the multiplier μ. Thus, finding the multipliers is equivalent to finding those values of μ for which the equation

$$x(t) = \Pi_\mu x(t) \tag{10}$$

has nonzero solutions. This latter problem can be solved approximately by means of projection methods (Ritz, Galerkin, Petrov, etc.), since the linear operator Π_μ is completely continuous in the spaces usually used, C, L_p, and others.

Let, for example, in $L_2[0, \omega]$ some complete system of linearly independent functions $g_1(t), g_2(t), \dots$ be chosen.

The approximate values of the multipliers are those values of μ for which the determinant of the linear homogeneous system

$$\int_0^\omega g_j(t)\,[\Pi_\mu(\xi_1 g_1 + \dots + \xi_n g_n) - \xi_1 g_1(t) - \dots - \xi_n g_n(t)]\,dt = 0$$

$$(j = 1, \dots, n) \tag{11}$$

is equal to zero. The corresponding functions $\xi_1 g_1(t) + \dots + \xi_{ng} n(t)$ are approximate solutions of system (7), and their initial values are approximate eigenvectors of the monodromy matrix. A detailed analysis of the proposed method has not been carried out. The question of the rate of convergence in the case of multiple multipliers is entirely unclear.

For the approximate computation of multipliers according to the described scheme, operators Π_μ different from (9) may be used, possessing the property that equation (10) has nonzero solutions only for those μ which are multipliers. For example, one could set

$$\Pi_\mu x(t) = \frac{1}{\mu} x(\omega) + \int_0^t B(s) x(s)\,ds \tag{12}$$

(however, the operator (12) can no longer be considered in the spaces L_p). It is clear how to write operator equations for finding the adjoint vectors corresponding to the multipliers.

3. Let us return to the study of the nonlinear system (1). We shall call the number λ_0 a bifurcation value of the parameter if, for every $\varepsilon > 0$, one can indicate a $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for which system (1) has at least one nonzero ω-periodic solution $x(t)$ whose amplitude $\max \|x(t)\|$ is less than ε.

Those values of the parameter λ for which one of the multipliers of system (7) is equal to 1 will be called critical. Obviously, *bifurcation values of the parameter can only be critical values.* The converse assertion is false in the general case, and it can be established only under additional assumptions.

Let λ_0 be an isolated critical value of the parameter. Denote by $\beta(\lambda_0 + 0)$ the sum of the multiplicities of the real multipliers greater than 1 of system (7) for values λ greater than λ_0, but close to λ_0. This sum, obviously, does not depend on λ. We define $\beta(\lambda_0 - 0)$ analogously.

It follows from general topological theorems (see (4)) that *the critical value λ_0 is a bifurcation value of the parameter if the sum $\beta(\lambda_0 - 0) + \beta(\lambda_0 + 0)$ is odd.* The case when $\beta(\lambda_0 - 0) = \beta(\lambda_0 + 0) \pmod 2$ is more complicated. In this case it is necessary to compute the index γ of the zero singular point of the vector field $x - A(x; \lambda_0)$ (see, for example, (4-7)). *If it turns out that $\gamma \neq (-1)^{\beta(\lambda_0 - 0)}$, then λ_0 is also a bifurcation value of the parameter.*

4. In what follows we consider the case when the isolated critical value λ_0 corresponds to a simple (nonmultiple) multiplier equal to 1. Then one may assume that in some neighborhood of the point λ_0 one of the multipliers $\mu_0(\lambda)$ is a single-valued and continuous function of λ, taking the value 1 only at the point λ_0, while the remaining multipliers take values different both from $\mu_0(\lambda)$ and from 1.

Denote by e_0 the unit eigenvector of the monodromy matrix ($U(\omega; \lambda_0)e_0 = e_0$), and by $x_0(t)$ the solution of system (7) (for $\lambda = \lambda_0$) satisfying the condition $x(0) = e_0$. The vector e_0 and the vector function $x_0(t)$ can be computed approximately, for example, by the method described in item 2. Denote by g_0 such a vector that $U^*(\omega; \lambda_0)g_0 = g_0$ and $(e_0, g_0) = 1$.

Since the right-hand sides of system (2) are sufficiently smooth, it can be written in the form

$$\frac{dx}{dt} = B(t; \lambda)x + C(t, x; \lambda) + D(t, x; \lambda), \tag{13}$$

where $C(t, x; \lambda)$ is an operator homogeneous in x of some order m, and $D(t, x; \lambda)$ consists of terms of higher order of smallness. Introduce the notation

$$k = \left(g_0, \int_0^\omega U(t; \lambda_0)U^{-1}(s; \lambda_0)C[s, x_0(s); \lambda_0]\, ds\right). \tag{14}$$

Theorem 1. *Let $k \neq 0$ and m be even. Then to every sufficiently small $\varepsilon_0 > 0$ there corresponds a $\delta > 0$ such that, for each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, $\lambda \neq \lambda_0$, system (1) has a unique nonzero ω-periodic solution whose amplitude is less than ε_0.*

Theorem 2. *Let $k \neq 0$ and m be odd. Then system (1) has no small nonzero ω-periodic solutions for those values of λ close to λ_0 for which*

$$\operatorname{sign}[\mu_0(\lambda) - 1] = \operatorname{sign} k, \tag{15}$$

and has exactly two small ω-periodic solutions for those values of λ close to λ_0 for which

$$\operatorname{sign}[\mu_0(\lambda) - 1] = -\operatorname{sign} k. \tag{16}$$

5. If the system (7) for $\lambda = \lambda_0$ has multipliers μ for which $|\mu| > 1$, then both the zero periodic solution of the system (1) and its small periodic solutions, which are discussed in Theorems 1 and 2, will be unstable.

The case of principal interest is when the system (7) satisfies the condition:

for $\lambda = \lambda_0$ all multipliers μ distinct from $\mu_0(\lambda)$ $(\mu_0(\lambda_0) = 1)$ satisfy the inequality $|\mu| < 1$. (*)

Theorem 3. *Suppose that the system (7) satisfies condition $(*)$. Suppose that the conditions of Theorem 1 are fulfilled. Then the small nonzero ω-periodic solution will be Lyapunov unstable for those values of λ close to λ_0 for which $\mu_0(\lambda) < 1$, and will be exponentially asymptotically stable for those values of λ close to λ_0 for which $\mu_0(\lambda) > 1$.*

Theorem 4. *Suppose that the system (7) satisfies condition $(*)$. Suppose that the conditions of Theorem 2 are fulfilled. Then the small nonzero ω-periodic solutions existing by virtue of Theorem 2 are unstable if $k > 0$, and exponentially asymptotically stable if $k < 0$.*

The case in which the system (7) for $\lambda = \lambda_0$ has multipliers μ distinct from 1 such that $|\mu| = 1$ is more complicated and requires a special analysis.

6. In conclusion I note that, for the proof of assertions of the type of Theorems 3 and 4, various methods may be applied: the method of Lyapunov functions, the theory of perturbations of linear operators, methods of nonlinear integral equations, etc. However, the methods of the theory of concave operators appear to us to be the simplest both from the conceptual and, especially, from the technical point of view.

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Note: Figure translations are in progress. See original paper for figures.