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Abstract

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MATHEMATICS

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UNIQUENESS OF THE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR A CONVEX DOMAIN

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The present article continues my preceding notes ([1,2]). In ([1]) a regular domain $G \subset R_n$ was defined. For it (in particular for $p = 2$) stable (more precisely, locally stable) boundary values were studied for a function Φ with finite integral

$$D_G(\Phi) = \int_G \sum_1^N |\Phi^{(k)}|^2 \, dG, \tag{1}$$

where

$$\mathbf{k}^1, \dots, \mathbf{k}^N \tag{2}$$

are given nonnegative vectors. Let the smallest convex body of the (integer) vectors spanned by the system (2), to which the vector 0 is adjoined, be $\mathcal{E}$. Together with $\mathbf{k}$, let $\mathcal{E}$ contain the projections of $\mathbf{k}$ onto coordinate subspaces of arbitrary dimensions. To a function Φ there is associated a definite set of boundary functions. This made it possible to speak of classes $\mathfrak{M} = \mathfrak{M}(G; \Phi)$ and $\mathfrak{M}_0 = \mathfrak{M}_0(G)$ of functions f with $D_G(f) < \infty$, possessing respectively the same sets as Φ and 0.

In ([2]) the following differential equation was considered in G:

$$Lu = \sum_{\mathbf{k},\mathbf{l} \in \mathcal{E}} (-1)^{(k)} D^{(k)}(\alpha_{\mathbf{kl}}(\bar{x}) u^{(l)}) = 0, \tag{3}$$

$$(\alpha_{\mathbf{kl}}(\bar{x}) = \alpha_{\mathbf{lk}}(\bar{x}), \ |\alpha_{\mathbf{kl}}(\bar{x})| \le M, \ x \in G),$$

for which, for example, it was assumed that

$$\sum_{\mathbf{k},\mathbf{l}\in\mathcal{E}} \alpha_{\mathbf{kl}}(\bar{x}) i^{|k|-|l|} \xi^{(k+l)} \geq \chi \sum_{1}^{N} \left(\xi^{(k^s)}\right)^2, \tag{4}$$

$$\xi^{(k)} = \xi_1^{k_1} \cdots \xi_n^{k_n}, \quad \mathbf{k} = (k_1, \dots, k_n), \quad |k| = \sum_{1}^{n} k_j,$$

where $\chi > 0$ does not depend on $\bar{x}$ and ξ. It was shown that for equation (3) there exists a unique generalized solution, and conditions were studied under which the existence of a classical solution of (3) is guaranteed. Equation (3) covers a broad class of equations of hypoelliptic (in particular elliptic) type, but in general goes beyond the limits of the hypoelliptic type.

The question arises of the uniqueness of a classical solution of (3). It is solved here in the case of a bounded convex domain G.

Let $\alpha_{\mathbf{kl}}$ be continuously differentiable in the domain G (open) $\mathbf{k}$ times, $\mathbf{k} \in \mathcal{E}$.

We shall say, for example, that u is a classical solution of (3) if u has in G continuous partial derivatives of orders $\mathbf{k}+\mathbf{l}$ ($\mathbf{k},\mathbf{l} \in \mathcal{E}$) and satisfies (3).

Theorem 1. A bounded convex domain G is regular. Consequently (see ([2])), in $\mathfrak{M}$ there exists, and moreover is unique, a generalized solution u of equation (3).

Theorem 2. Let Φ have a finite integral and, for every $\mathbf{k} \in \mathcal{E}$ for which $\alpha_{k1}(x) \not\equiv 0$, let $\Phi^{(\mathbf{k})} \in L_2(G)$, where G is a bounded convex domain. Then in the class $\mathfrak{M} = \mathfrak{M}(G;\Phi)$ there can exist one classical solution of equation (3).

In the proof of the theorems one introduces $\widetilde{G}_s^j$—the projection of G onto some (with number s) coordinate plane $R_{n-j}^{(s)}$ of dimension $n-j$ ($j = 0, 1, \dots, n-1$); $\widetilde{G}_{s,\delta}^j$ is the set of $\bar{x} \in \widetilde{G}_s^j$ lying at a distance greater than $\delta > 0$ from the boundary of $\widetilde{G}_s^j$; $G_{s,\delta}^j$ is the cylindrical body constructed on $\widetilde{G}_{s,\delta}^j$ with generators belonging to $R_j^{(s)}$, where $R_n = R_j^{(s)} \times R_{n-j}^{(s)}$. Here $\widetilde{G}_1^0 = G_1^0 = G$, $\widetilde{G}_{1,\delta}^0 = G_{1,\delta}^0 = G_\delta$—the set of $\bar{x} \in G$ lying at a distance greater than δ from the boundary Γ of the domain G. The corresponding cylinder constructed on G_δ coincides with G_δ (here only $s = 1$ is possible).

Theorem 1 follows for $n = 3$ from the following facts. Let Γ_s^1 and Γ_s^2 ($s = 1, 2, 3$) be the intersections of Γ, respectively, with the boundary of G_s^1, G_s^2. The points $\Gamma - \bigcup_{s=1}^3 \Gamma_s^1$ and the interior (with respect to Γ_s^1, Γ_s^2) points of Γ_s^1, Γ_s^2 are regular. For any $s = 1, 2, 3$ the edge (boundary of a two-dimensional surface) Γ_s^1, Γ_s^2 has projection onto any plane $x_j = 0$ of two-dimensional measure zero.

Lines passing through the points $\bar{x} \in G_s^1$ intersect Γ in two points—the upper and the lower (since G is open, so are $\widetilde{G}_s^j$, G_s^j). The upper points form the set Γ_s^1, the lower the set Γ_s^0. Assign positive numbers $\delta, \sigma_1, \dots, \sigma_n$ and put $\omega_s^0 = \Gamma_s^0 G_{s,\sigma_s}^{0*}$,

$\omega_s^1 = \Gamma_s^1 G_{s,\sigma_s}^{1*}$ $(2\sigma_s^* = \sigma_s)$. The sets ω_s^0, ω_s^1 are at a distance greater than some positive number. Let $\Omega_s = (G - G_{s,[\delta]})G_{s,\sigma_s}^1$. For sufficiently small δ_0, depending on σ_s, Ω_s for $0 < \delta < \delta_0$ decomposes into two nonintersecting sets $\Omega_s = \Omega_s^0 + \Omega_s^1$, adjacent respectively to ω_s^0, ω_s^1. Introduce the sets $\omega_{i_1,\dots,i_n} = \omega_1^{i_1} \cdots \omega_n^{i_n}$, $\Omega_{i_1,\dots,i_n} = \Omega_1^{i_1} \cdots \Omega_n^{i_n}$, where $i_s = 0, 1$. There are decompositions into sums of pairwise nonintersecting sets

$$\omega = \Gamma G'_{1,\sigma_1} \cdots G'_{n,\sigma_n} = \sum_{i_s=0,1} \omega_{i_1,\dots,i_n}$$

and, for sufficiently small δ_0, depending on $\sigma_1, \dots, \sigma_n$,

$$\Omega^0 = (G^0 - G_{3\delta}^0)G'_{1,\sigma_1} \cdots G'_{n,\sigma_n} = \sum \Omega_{i_1,\dots,i_n} \qquad (0 < \delta < \delta_0). \tag{5}$$

$$G - G_{3\delta} = G_1^0 - G_{1\delta}^0.$$

For a function $v \in \mathfrak{M}_0(G)$ and $\mathbf{k} \ll 1 \in \mathcal{E}$, the basic inequality is proved:

$$\left(\int_{\Omega_0} |v^{(1-\mathbf{k})}|^2 \, dG\right)^{1/2} \le \sum_{\Omega_{i_1,\dots,i_n}} \left(\int |v^{(1-\mathbf{k})}|^2 \, dG\right)^{1/2} \le$$

$$\le c\delta^{|\mathbf{k}|} \left(\int_G |v^{(1)}|^2 \, dG\right)^{1/2} \qquad (0 < \delta < \delta_0), \tag{6}$$

where c does not depend on δ and v.

Let us now take as the initial set, instead of G, a certain $\widetilde{G}_s^j$. It belongs to some $R_{n-j}^{(s)}$ (depending on s). We shall denote its projections onto the coordinate $(n-j-1)$-dimensional subspaces $R_{n-j}^{(s)}$ by $\widetilde{G}_{s,j+1}^{j+1}, \dots, \widetilde{G}_{s,n}^{j+1}$ (with indices $j+1, \dots, n$). Let

$$\Omega_s^j = (G_s^j - G_{s,3\delta}^j) G_{(s,j+1),\sigma_{j+1}}^{j+1} \cdots G_{(s,n),\sigma_n}^{j+1},$$

$$R_n = R_j^{(s)} \times R_{n-j}^{(s)}$$

and let $\mathbf{k} \le \mathbf{1} \in \mathcal{E}$, and all projections k_i of the vector $\mathbf{k}$ onto $R_j^{(s)}$ be equal to zero. Then, if $v \in \mathfrak{M}_0(G)$, the following estimate, more general than (6), holds:

$$\int_{\Omega_s^j} |v^{(\mathbf{1}-\mathbf{k})}|^2 \, dG \le c\delta^{|\mathbf{k}|} \int_G |v^{(\mathbf{1})}|^2 \, dG \qquad (0 < \delta < \delta_0;\ j = 0, 1, \dots, n-1), \tag{7}$$

where c does not depend on δ or v.

Let now $u \in \mathfrak{M}$ be a classical solution of equation (3) in G. If it is proved that u is a generalized solution of (3), i.e. that $E_G(u,v)=0$ (see (2)) for all $v \in \mathfrak{M}_0$, then this will prove uniqueness, since the generalized solution is unique.

Restricting ourselves to the three-dimensional case, introduce the positive numbers $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_1^2, \sigma_2^2, \sigma_3^2$ and the functions $\alpha^0_{\sigma_0}, \alpha'_{1,\sigma_1}, \alpha'_{2,\sigma_2}, \alpha'_{3,\sigma_3}, \alpha^2_{1,\sigma_1^2}, \alpha^2_{2,\sigma_2^2}, \alpha^2_{3,\sigma_3^2}$, depending on $\bar{x}$, equal respectively to 1 on $G_{2\sigma_0}, G'_{2\sigma_1}, \dots, G^2_{2\sigma_3^2}$ and to 0 outside these sets. Let the functions of $\bar{x}$ $\eta=\eta^0_{\sigma_0}, \eta_1=\eta_{\sigma_1}, \dots, \eta^2_{3,\sigma_3^2}$ be respectively their $\sigma_0, \dots, \sigma_3^2$-Sobolev averages (see (3)). Let $v \in \mathfrak{M}_0$ and

$$v_0 = \eta\eta_1\eta_2\eta_3\eta_1^2\eta_2^2\eta_3^2 = \eta v_1 = \eta\eta_1 v_2 = \cdots$$

not only belong to $\mathfrak{M}_0$, but also be equal to zero along the strip in G adjacent to Γ. Therefore $E_G(u,v_0)=0$, and, taking into account that $\eta=\eta_1=\dots=\eta_3^2=1$ on

$$Q_0 = G_{3\sigma_0}G_{1,3\sigma_1}\cdots G^2_{3,3\sigma_3^2} = G_{3\sigma_0}Q_1 = G_{3\sigma_0}G_{1,3\sigma_1}Q_2 = \cdots,$$

one may easily conclude that uniqueness will be proved if it is proved that

$$\overline{\lim_{\sigma_3^2\to 0}}\;\overline{\lim_{\sigma_2^2\to 0}}\cdots\overline{\lim_{\sigma_0\to 0}} E_{G-Q_0}(u,v_0)=0.$$

This method, in the simplest case, when $v_0=\eta_\delta v$, was applied by S. L. Sobolev (3) to prove uniqueness of the solution of the polyharmonic problem. In the present complicated case one has to pass to the limit successively $n+1$ times. Directly, by a single passage to the limit, the desired property cannot be detected.

From the condition imposed in Theorem 2 on Φ it follows that u also possesses this property. This leads to the fact that, for uniqueness, it is sufficient to prove that

$$\overline{\lim_{\sigma_3^2\to 0}}\cdots\overline{\lim_{\sigma_0\to 0}}\int_{G-Q_0}|v_0^{(\mathbf{1})}|^2\,dG = A < \infty.$$

We have

$$\left(\int_{G-Q_0}|v_0^{(\mathbf{1})}|^2\,dG\right)^{1/2} \le c\sum_{k\le\mathbf{1}}\left(\int_{(G-G_{3\sigma_0})G_{1,\sigma_1}G_{2,\sigma_2}G_{3,\sigma_3}}|\eta^{(\mathbf{k})}v_1^{(\mathbf{1-k})}|^2\,dG\right)^{1/2}+$$

$$+\left(\int_{(G-Q_0)G_{3\sigma_0}} |v_1^{(\mathbf{1})}|^2\,dG\right)^{1/2},$$

and since $|\eta^{(k)}| \ll c\sigma_0^{-|k|}$, it follows from (7), for $j=0$, that the sum $\sum_{k\leqslant l}$ is bounded, whence

$$\overline{\lim_{\sigma_0\to 0}}\int_{G-Q_0} |v_0^{(1)}|^2\,dG = \int_{G-Q_1} |v_1^{(1)}|^2\,dG + B_1, \qquad B_1<\infty.$$

Further,

$$\int_{G-Q_1} |v_1^{(1)}|^2\,dG \ll c\sum_{k\leqslant l}\left(\int_{(G-G_{1,3\sigma_1})G^2_{1,\sigma_1}G^2_{2,\sigma_2}G^2_{3,\sigma_3}} |\eta_1^{(k)}v_2^{(1-k)}|^2\,dG\right)^{1/2} + \left(\int_{(G-Q_1)G_{1,3\sigma_1}} |v_2^{(1)}|^2\,dG\right)^{1/2}. \tag{8}$$

The function η_1 does not depend on x_1; therefore, if $k=(k_1,\dots,k_n)$ and $k_1>0$, then $\eta_1^{(k)}\equiv 0$; but if $k_1=0$, then, taking into account that

$$(G-G_{1,3\sigma_1})G^2_{1,\sigma_1}G^2_{2,\sigma_2}G^2_{3,\sigma_3} \subset (G_1'-G'_{1,3\sigma_1})\widetilde{G}^2_{2,\sigma_2}\widetilde{G}^2_{3,\sigma_3}$$

(G_2^2, G_3^2 are cylinders constructed on $\widetilde{G}_2^2$, $\widetilde{G}_3^2$, the projections of $\widetilde{G}_1$ onto the axes x_2, x_3), then, by virtue of the inequality $|\eta_1^{(k)}| \ll c\sigma_1^{|k|}$ and (7) for $j=1$, we conclude that the sum $\sum_{k\leqslant l}$ is bounded, while the second term on the right-hand side of (8), as $\sigma_1\to 0$, tends to

$$\left(\int_{G-Q_2} |v_2^{(1)}|^2\,dG\right)^{1/2}.$$

We reason in the same spirit in the cases where $\sigma_2, \sigma_3, \dots$ tend to zero. Here one must take into account that each of the functions $\eta_1^2, \eta_2^2, \eta_3^2$ does not depend on two variables, and in the corresponding place (7) is applied with $j=2$.

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Note: Figure translations are in progress. See original paper for figures.