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Abstract

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MATHEMATICS

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PRECLOSED MAPPINGS AND A THEOREM OF A. D. TAIMANOV

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Several assertions are known which are valid separately both for closed and for open mappings. For example, the theorem on the connectedness of the complete inverse image of a connected set and on the zero-dimensionality of the image of a zero-dimensional set under monotone mappings. It is therefore very tempting to find a class of mappings, including both closed and open ones, for which these assertions would remain true.

One attempt of this kind was made long ago by G. T. Whyburn ([2]), who introduced the concept of a quasi-compact mapping. However, the propositions proved by him are true under certain restrictions, which it would be desirable to dispense with. Here another attempt is made. Yu. M. Smirnov pointed out to me the possibility of another definition of the desired class of mappings—we call them **preclosed***(see Definition 1). For them not only the propositions indicated above** (see Theorem 2) turn out to be true, but also the remarkable theorem of A. D. Taimanov*** on the extension of monotone mappings to monotone mappings on the Čech extension; moreover, instead of Čech extensions we consider any perfect extensions (in the sense of E. G. Sklyarenko ([3])) and even not necessarily bicompact ones (Theorem 5). In a certain sense a converse Theorem 6 is also proved. In connection with this we obtain here some characteristics of the "perfection" of an extension, different from those proposed by E. G. Sklyarenko (Theorem 7).

Of course, at the beginning we find conditions under which the properties of quasi-compactness and preclosedness coincide—in any case, this is so for spaces with the first axiom of countability (Theorem 1). In addition, several topological properties are given which are preserved under bicompact preclosed mappings (Theorems 3 and 4).

Lemma 1. Let f be a mapping**** of a space X onto a space Y; let O be a neighborhood of the complete inverse image $f^{-1}y$ of a point y, $y \in Y$. The following two properties a) and b) are equivalent: a) there exists a set H such that $f^{-1}y \subset H \subset O$ and its image fH is open; b) there exists a neighborhood V_y of the point y such that

$$f[O \cap f^{-1}(V_y)] = V_y.$$

Definition 1. A mapping f of a space X onto a space Y will be called **preclosed** if, for every point y of Y and for every neighborhood O of its complete inverse image, condition a) or b) is fulfilled.

Remark 1. Every open and every closed mapping is preclosed.

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* The name is conditional: they are as preclosed as they are preopen.

** As A. V. Arkhangel' skii informed me, one of his theorems, earlier proved by him separately for open and separately for closed mappings (see Theorem 12 ([1])), turned out to be true also for preclosed mappings.

*** See ([4]); G. T. Whyburn did not consider this theorem, since it was obtained by A. D. Taimanov considerably later.

**** By a mapping we agree to understand here only a single-valued mapping, and by a space—a topological space.

Lemma 2. If a mapping f of a space X onto a space Y is preclosed, then it is preclosed on every inverse* subset of the space X.

Lemma 3. A mapping f of a space X onto a space Y is preclosed if it is preclosed on at least one such set A, $A \subset X$, that $fA = Y$.

Lemma 4. Every preclosed mapping is quasicompact.**

Theorem 1. A continuous mapping of a space X onto a Hausdorff space Y, satisfying the following condition b) (weaker than the first axiom of countability), is preclosed if and only if it is quasicompact: b) for every point y of the space Y there is such a countable sequence of neighborhoods $\{O_k y\}$ that, if $y_k \in O_k y$, then the sequence $\{y_k\}$ converges to the point y.

Example 1. A continuous monotone*** quasicompact, but not preclosed, mapping g. Let X' be the number line, and let $g'x = x$ if x is not a positive integer, and $g'x = -x^{-1}$ otherwise. In the space $Y = g'X'$, a set M is regarded as open if and only if the set ${g'}^{-1}M$ is open. The space Y turns out to be normal, hereditarily finally compact, and does not satisfy condition b)****. Adding in the space E^3, to each pair of points $\{k, -k^{-1}\}$, the arc $C_k = [k, -k^{-1}]$, and putting $gx = g'x$ if $x \in X'$, and $gx = -k^{-1}$ if $x \in C_k$, with a proper arrangement of these arcs, we obtain the required space X and the mapping g.

Theorem 2. Let a mapping f of a space X onto a space Y be monotone and preclosed; then the complete preimage of every connected set is connected, and the image of every null-dimensional***** inverse set is null-dimensional, if one additionally assumes that f is continuous.

Remark 2. The assertions of Theorem 2 for quasicompact mappings are, generally speaking, not true******. The first is true for the entire space Y, and the second for X*******.

Theorem 3. Under a continuous, preclosed, and bicompact******** mapping, the weight of a space (if it is infinite) cannot increase.

Theorem 4. Under continuous, preclosed, and $[a, \infty]$-compact mappings (if the cardinality α is infinite!) the property of local $[a, b]$-compactness is preserved.

Definition 2. Following E. G. Sklyarenko ([3]), we shall call an extension********* cY of a space Y **perfect** if, for any two closed sets A and B of the space Y such that $A \cup B = Y$, the equality

$$\overline{A} \cap \overline{B^c} = \overline{A^c} \cap \overline{B^c}. \tag{1}$$

holds.

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* A subset A of the space X is called **inverse** if $A = f^{-1} f A$.

** Following G. T. Whyburn ([2]), we call a mapping **quasicompact** if the image of every inverse open set is open.

*** A mapping f is called **monotone** if every complete preimage $f^{-1} y$ is connected.

**** It is very similar to a "hedgehog."

***** Null-dimensionality here may be understood in any sense: ind, Ind, or dim.

****** This can be shown by slightly modifying Example 1.

******* This follows from Lemmas 2 and 4 of Theorem 2.

******** We shall call a mapping $[a, b]$-**compact** if every complete preimage is $[a, b]$-compact (i.e., from each of its open covers of cardinality $\leq b$ one can choose a subcover of cardinality $< a$). $[a, \infty]$-**compactness** is $[a, b]$-compactness for b equal to the cardinality of the space under consideration.

********* E. G. Sklyarenko considered only bicompact extensions of completely regular spaces. In the course of the proof we shall need Hausdorff extensions and spaces.

Theorem 5. Let aX be an extension* of the space X, and let f be a preclosed mapping of the space X onto the space Y, extendable to a perfect** mapping f_{ac} of the extension aX onto some perfect extension cY of the space Y; if f is monotone, then f_{ac} is monotone***.

Proof. Suppose that the hypotheses of the theorem are satisfied, but f_{ac} is not monotone. Then there exists a point $y \in cY$ with disconnected inverse image $f_{ac}^{-1}y$. Hence $f_{ac}^{-1}y = A \cup B$, where A and B are nonempty disjoint bicompacts. Therefore there are open sets U and V of the extension aX such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. Since $\overline{f(X \setminus U)} \cup \overline{f(X \setminus V)} = Y$****, we have

$$\overline{f(X \setminus U)}^c \cap \overline{f(X \setminus V)}^c = \overline{\overline{f(X \setminus U)} \cap \overline{f(X \setminus V)}}^c.$$

Since $A \subseteq \overline{U}^a = \overline{U} \cap X^a \subseteq \overline{X \setminus V}^a$, it follows that

$$y \in \overline{f(X \setminus V)}^c.$$

Similarly, $y \in \overline{f(X \setminus U)}^c$. Thus,

$$y \in \overline{f(X \setminus U)}^c \cap \overline{f(X \setminus V)}^c.$$

But f_{ac} is closed and $f_{ac}^{-1}y \subseteq U \cup V$. Hence there exists a neighborhood H of the point y satisfying the inclusion $f_{ac}^{-1}H \subseteq U \cup V$. The set

$$H' = H \cap \overline{f(X \setminus U)}^c \cap \overline{f(X \setminus V)}^c$$

is nonempty. Let $y' \in H'$. Then $f^{-1}y' \subseteq U \cup V$, and therefore either $f^{-1}y' \subseteq U$, or $f^{-1}y' \subseteq V$. Suppose, for example, that $f^{-1}y' \subseteq U$. By preclosedness there exists a set W whose image fW is open in Y, such that $f^{-1}y' \subseteq W \subseteq U \cap X$. In view of the special choice of the point y', the set

$$H'' = H \cap fW \cap \overline{f(X \setminus U)}$$

is nonempty. Let $y'' \in H''$. Again we have $f^{-1}y'' \subseteq U \cup V$. Moreover, $f^{-1}y'' \cap W \neq \emptyset$, and hence $f^{-1}y'' \cap U \neq \emptyset$. On the other hand, $f^{-1}y'' \cap (X \setminus U) \neq \emptyset$. This contradicts the connectedness of the inverse image $f^{-1}y''$. The theorem is proved.

Theorem 6. Let aX be a perfect extension of the space X, and let f be a mapping of the space X onto the space Y, extendable to a closed mapping f_{ac} of the extension aX onto some extension cY of the space Y; if the mapping f_{ac} is monotone, then the extension cY is perfect*****.

Lemma 5. A mapping g of the space X onto the space Y is continuous and closed if and only if it commutes with the operation of closure: $g\overline{A} = \overline{gA}$ for all A in X.

Proof of the theorem. Suppose that cY is not perfect, i.e., there exist closed sets A and B in Y such that $A \cup B = Y$, but $\overline{A}^c \cap \overline{B}^c \neq \overline{A \cap B}^c$. It is clear that

$$\overline{f^{-1}A}^a \cap \overline{f^{-1}B}^a = \overline{f^{-1}A \cap f^{-1}B}^a.$$

Let

$$y \in \overline{A}^c \cap \overline{B}^c \setminus \overline{A \cap B}^c.$$

Since

$$f_{ac}\overline{f^{-1}A}^a = \overline{A}^c,$$

we have

$$f_{ac}^{-1}y \cap \overline{f^{-1}A}^a \neq \emptyset.$$

Similarly,

$$f_{ac}^{-1}y \cap \overline{f^{-1}B}^a \neq \emptyset.$$

Since $y \notin \overline{A \cap B}^c$, for the same reason we have

$$f_{ac}^{-1}y \cap \overline{f^{-1}(A \cap B)}^a = \emptyset,$$

and therefore

$$f_{ac}^{-1}y \cap \overline{f^{-1}A}^a \cap \overline{f^{-1}B}^a = \emptyset.$$

This contradicts the connectedness of the inverse image $f_{ac}^{-1}y$. The theorem is proved.

* Beginning from this point, we shall consider only Hausdorff extensions (and spaces!) and only continuous mappings.

** A mapping is called **perfect** if it is closed and if the full inverse images of all points of the space Y are bicompact.

*** This is the generalized A. D. Taimanov theorem. The extensions aX and cY in his formulation were bicompact; moreover, cY was the Čech extension (the spaces X and Y, consequently, completely regular). Here the extensions and spaces are only Hausdorff.

**** $\overline{M}$ is the closure of the set M, $M \subseteq Y$, in the space Y, and $\overline{M}^c$ is the closure of the set M in the extension cY.

***** In this theorem, as also in Lemma 5, the spaces X and Y, as well as their extensions, may be considered without any restrictions.

Lemma 6. The Čech extension βY of every completely regular space Y is perfect. Every subset bY of the perfect extension cY, containing Y, is also perfect.

For brevity, in what follows we shall call extensions bY such that $\beta Y \supseteq bY \supseteq Y$ **Čech** extensions*.

Corollary 1. An extension cY of a space Y is perfect if and only if it is the image of some perfect extension aX of some X under a monotone and perfect** mapping f_{ac} such that $f_{ac}X = Y$, and such that it is monotone and preclosed on X.

Corollary 2. A completely regular extension cY of a space Y is perfect if and only if it is a monotone and perfect image of some Čech extension***.

Remark. It is not hard to see that in Theorem 5 the condition of perfection of the extension cY in the case when the extension aX is strongly Hausdorff**** can be replaced, with the same result, by either of the following two conditions:

A. For any two such closed sets A and B that $\langle f^{-1}A\rangle \cup \langle f^{-1}B\rangle = X$, equality (1) of Definition 2 is always true.

B. For any two such closed sets A' and B' of the space X such that $\bar{A'}^{a} \cap \bar{B'}^{a} = \emptyset$ for the sets $A = f\bar{A}'$ and $B = f\bar{B}'$, equality (1) of Definition 2 holds.

Theorem 7. If the extension cY is completely regular, then each of the following conditions (separately) is necessary and sufficient in order that the extension cY be perfect: 1) there exist a space X and a mapping $f : X \to Y$ such that condition A is fulfilled (see above); 2) there exist a space X and a mapping $f : X \to Y$ such that, for any two such closed sets A and B of the space Y for which $f^{-1}A \cup f^{-1}B = X$, equality (1) of Definition 2 is true; 3) for any two such closed sets A and B of the space Y for which $\langle A\rangle \cup \langle B\rangle = Y$, equality (1) of Definition 2***** is true.

Proof. Indeed, perfection implies condition 2), from it condition 1), and from 1) condition 3). It is not hard to see that the extension cY is a perfect image of some Čech extension bY. In this case condition 3) is identical with condition 1), whence, by virtue only of the remark just made, and also by Corollary 3, the perfection of the extension cY follows.

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CITED LITERATURE

1. A. V. Arkhangel' skii, DAN, 147, No. 5, 999 (1962).
2. G. T. Whyburn, Duke Math. J., 17, No. 1, 69 (1950).
3. E. G. Sklyarenko, DAN, 137, No. 1, 39 (1961).
4. A. D. Taimanov, DAN, 135, No. 1, 23 (1960).

* The "true" Čech extension βY is distinguished among them by the fact that it alone is bicompact.

** In the case of bicompactness of the extension cY, the condition of perfection of the mapping is equivalent to the condition of bicompactness of the extension aX.

*** In the case of bicompactness of the extension cY, this corollary is one of the theorems of E. G. Sklyarenko.

**** A space (respectively, an extension) is called **strongly Hausdorff** if any two of its points have disjoint closed neighborhoods.

***** Condition 3) is perhaps the most curious one. It is easy to see that complete regularity of the extension cY (and, consequently, of the space Y) is not essential here: it is enough to require that there exist a strongly Hausdorff perfect extension bY of the space Y such that the identity mapping e could be extended to a perfect mapping of the extension bY onto cY.

Note: Figure translations are in progress. See original paper for figures.