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Abstract

Full Text

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ON A. WALD' S TEST

In the well-known paper ⁽¹⁾, A. Wald considers the Behrens-Fisher problem for two independent samples $x_{11}, \dots, x_{1n_1}; x_{21}, \dots, x_{2n_1}$ of equal size n_1 ; here $x_{1j} \in N(a_1, \sigma_1)$, $x_{2j} \in N(a_2, \sigma_2)$ are repeated normal samples. Introducing four sufficient statistics:

$$\bar{x}_i = \frac{1}{n_1} \sum_{j=1}^{n_1} x_{ij}$$

$$(i = 1, 2); \quad s_i^2 = \frac{1}{n_1 - 1} \sum_{j=1}^{n_1} (x_{ij} - \bar{x}_i)^2,$$

A. Wald then postulates four requirements for the critical region of the (nonrandomized) test under consideration:

- I. The critical region lies in the space of the sufficient statistics $\bar{x}_1, \bar{x}_2, s_1^2, s_2^2$.
- II. If the sample $(x_{11}, \dots, x_{1n_1}; x_{21}, \dots, x_{2n_1})$ lies in the critical region, then for any c the sample $(x_{11} + c, \dots, x_{1n_1} + c; x_{21} + c, \dots, x_{2n_1} + c)$ also lies there.
- III. If the sample $(x_{11}, \dots, x_{1n_1}; x_{21}, \dots, x_{2n_1})$ lies in the critical region, then for $k \neq 0$ the sample $(kx_{11}, \dots, kx_{1n_1}; kx_{21}, \dots, kx_{2n_1})$ also lies there.
- IV. If the sample $(x_{11}, \dots, x_{1n_1}; x_{21}, \dots, x_{2n_1})$ lies in the critical region and for a second sample $(x'_{11}, \dots, x'_{1n_1}; x'_{21}, \dots, x'_{2n_1})$ we have

$$|\bar{x}'_2 - \bar{x}'_1| > |\bar{x}_2 - \bar{x}_1|, \quad s'^2_1 = s^2_1, \quad s'^2_2 = s^2_2,$$

then the second sample also lies there.

From these conditions it is not difficult to derive that the critical region has the form

$$\frac{|\bar{x}_2 - \bar{x}_1|}{\sqrt{s_1^2 + s_2^2}} \geq \phi(\eta); \quad \eta = \frac{s_2^2}{s_1^2}, \quad (1)$$

where $\phi(\eta)$ is a single-valued function of η . The quantity η has the distribution λF , where $\lambda = \sigma_2^2/\sigma_1^2$, and F is a random variable having on the axis $(0, \infty)$ Fisher' s distribution $F_{n_1 n_1}$. A. Wald replaces the distribution F by the distribution F^* , where, for a given $F_0 > 0$,

$$P(F^* < u) = P(F < u) \quad \text{for } u < F_0,$$

$$P(F^* < u) = 1 \quad \text{for } u > F_0$$

(“atomization” of the point F_0). With this new distribution λF^* (instead of λF) for the quantity η , it is possible to construct analytically in a neighborhood $0 \leq |\eta| \leq \eta_0 = \eta_0(F_0)$ a function $\phi(\eta)$ for which the critical region (1) will be similar for $0 \leq \lambda \leq \lambda_0 = \lambda_0(F_0)$. To this end, an algorithm is given for constructing the series

$$\phi(\eta) = c_0 + c_1\eta + c_2\eta^2 + \dots$$

and the convergence of the series for $|\eta| \leq \eta_0$ and the similarity of the region (1) for $0 \leq \lambda \leq \lambda_0$ are proved. This gives an approximate similar test for the original situation.

A. Wald raises the question whether such a function exists also in the original situation (without replacing F by F^*). In the present note a negative answer to this question is given.

There does not exist a piecewise-analytic function $\phi(\eta)$ for which the critical region (1) would be similar.

We shall formulate a stronger theorem concerning this question. Put

$$\frac{\bar{x}_1 - \bar{x}_2}{s_1} = \xi; \quad \phi(\eta)\sqrt{1 + \eta} = \Phi(\eta).$$

Then the critical region (1) takes the form

$$|\xi| \geq \Phi(\eta),$$

where $\Phi(\eta)$ is a single-valued measurable function of η . It is possible to consider quadrant $\xi \geq 0$, $\eta \geq 0$, and in it the critical region

$$\xi \geq \Phi(\eta) \tag{2}$$

with boundary $\Gamma : \xi = \Phi(\eta)$.

Suppose that this boundary is continuous in a neighborhood of $\eta = 0$. Then, as it turns out, the boundary Γ intersects the axis of abscissas at the point $(\xi_0, 0)$, where $\xi_0 < \infty$, and if the test is nontrivial, then $\xi_0 > 0$. Let $\varepsilon_0 > 0$ be an arbitrarily small prescribed number. Consider the region in the form of a “half-horseshoe” Π between quadrants of confocal ellipses:

$$\frac{\xi^2}{\xi_0^2 \pm \varepsilon_0} + \frac{\eta^2}{\xi_0^2 + 1 \pm \varepsilon_0} = 1; \quad \xi \geq 0; \quad \eta \geq 0.$$

Theorem. *Such a test of A. Wald (2) is impossible if its boundary inside the half-horseshoe Π consists of a finite number of pieces of curves differentiable $n_1 + 1$ times.*

Let us note that the same result on the impossibility of the test is obtained if similarity of the test is assumed not for all values $\lambda = \xi_2^2/\sigma_1^2$, but only for some countable set of different values of λ lying in a bounded interval.

We shall briefly explain the method of proof of the theorem. It is based on introducing an analytic continuation with respect to the parameter (see the general considerations on this subject in ²). Put: $g(\xi, \eta) = 1$ for $\xi \geq \Phi(\eta)$; $g(\xi, \eta) = 0$ otherwise. Then the following equation is obtained:

$$\iint_{\Omega_1} g(\xi, \eta) \frac{\eta^{n_1-2} d\xi d\eta}{(\vartheta + A(\xi, \eta))(\vartheta + B(\xi, \eta))^N} = A_{n_1} \vartheta^{-n_1/2} (1 + \vartheta)^{-n_1-1}, \quad (3)$$

where Ω_1 is the quadrant $\xi \geq 0, \eta \geq 0$; $\vartheta \geq 0$ is a parameter; $N = n_1 - \frac{1}{2}$; $A_{n_1} > 0$ is a constant, and $A(\xi, \eta)$ and $B(\xi, \eta)$ are the roots, taken with the opposite sign, of the equation

$$\vartheta^2 + \vartheta(1 + \xi^2 + \eta^2) + \eta^2 = 0. \quad (4)$$

The notation $A(\xi, \eta)$ and $B(\xi, \eta)$ may be chosen so that always: $0 \leq A(\xi, \eta) \leq 1$; $B(\xi, \eta) \geq 1$. The curves $B(\xi, \eta) = D_0$ for $D_0 > 1$ are confocal ellipses

$$\frac{\xi^2}{D_0 - 1} + \frac{\eta^2}{D_0} = 1;$$

the curves $A(\xi, \eta) = D_0$ for $D_0 < 1$ form a family of hyperbolas

$$\frac{\eta^2}{D_0} - \frac{\xi^2}{1 - D_0} = 1.$$

These curves will be the “critics” of the family of measures. If $D_0 = \xi_0^2 + 1$ is chosen, then the arc of the ellipse $B(\xi, \eta) = D_0$ will issue from the axis of abscissas together with the test boundary Γ . Relation (3) is analytically continued in ϑ onto the plane of complex values of ϑ , cut along the negative axis: $\text{Im } \vartheta = 0, -\infty < \text{Re } \vartheta \leq 0$. If we put $\vartheta = \vartheta_0 = -(\xi_0^2 + 1) + i\zeta$, where $\zeta > 0$ is a small number, then relation (3) will be valid. If it is differentiated twice with respect to ϑ at the point $\vartheta = \vartheta_0$ and then $\zeta \rightarrow 0$ is considered, the right-hand side of (3) will approach a finite limit, while the modulus of the left-hand side will tend to ∞ (from (3) it is seen that in the denominator of the integrand there appears a factor close to $(i\zeta)^N$), which leads to a contradiction. These arguments can be carried out rigorously using the conditions of the theorem.

The piecewise differentiability of the test boundary $n_1 + 1$ times inside the half-horseshoe Π is needed in order to study possible intersections and tangencies of the boundary Γ and the critic $B(\xi, \eta) = \xi_0^2 + 1$.

The conditions of the theorem can be weakened still further.

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References

1. A. Wald, *Selected Papers in Probability and Statistics*, N. Y., 1955, p. 669.
2. Yu. V. Linnik, DAN, **149**, No. 5 (1963).

Note: Figure translations are in progress. See original paper for figures.

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