V. D. Tretyakov

1963

SovietRxiv

Abstract

Full Text

V. D. Tretyakov

On the Question of Line Geometry in Three-Dimensional Klein Spaces

(Presented by Academician A. I. Mal' tsev on 27 IV 1963)

In the present paper a correspondence is established between symmetric conformally Euclidean spaces (SC-spaces) of four dimensions and zero signature and the line geometries of various subgroups of projective transformations of three-dimensional space.

1. Consider the collineation

$$\tilde{x}^{\alpha} = \gamma^{\alpha}_{\sigma} x^{\sigma} \qquad (\alpha, \beta, \dots, \sigma = 1, 2, \dots, n+2); \tag{1}$$

in the projective space P_{n+1}, satisfying the condition

$$\gamma^{\alpha}_{\sigma} \gamma^{\sigma}_{\beta} = \varepsilon \delta^{\alpha}_{\beta} \qquad (\varepsilon = \pm 1,\ 0). \tag{2}$$

We shall call this collineation absolute.

The generalized biplanar space B_{n+1} is the $(n+1)$-dimensional space whose fundamental group is isomorphic to the subgroup of projective transformations taking the absolute collineation (1) into itself.

For $\varepsilon = 1$ (hyperbolic B_{n+1}) the absolute collineation is a projective symmetry in an m-pair; the case $\varepsilon = -1$ is possible only in a space of an odd number of dimensions (elliptic B_{2k+1}) and corresponds to a biplanar involution of elliptic type; for $\varepsilon = 0$ (parabolic B_{n+1}) the matrix $(\gamma^{\alpha}_{\beta})$ is a nilpotent matrix of general form ([6], *p.* 16).

A B-quadric Q_n of the generalized biplanar space B_{n+1} is called ([6], *p.* 17) a quadric Q_n, $a_{\alpha\beta} x^{\alpha} x^{\beta} = 0$, defined by a symmetric tensor $a_{\alpha\beta}$, for which the tensor adjoint to it ([2], *p.* 145)

$$b_{\alpha\beta} = a_{\alpha\sigma} \gamma^{\sigma}_{\beta} \tag{3}$$

is also symmetric.

The absolute planes of the collineation (1) are polar conjugate with respect to the B-quadric in the elliptic and hyperbolic cases and belong to it in the parabolic case.

2. B. A. Rosenfeld has proved ([3], *p.* 368) that the groups of motions of the spaces of constant curvature S_3, ${}^1S_3$, and ${}^2S_3$ are isomorphic to subgroups of motions of the space ${}^3S_5$ leaving fixed two planes that are polar conjugate with respect to the absolute, i.e., taking into themselves certain involutions in P_5. Thus, these groups are isomorphic to subgroups of biplanar motions preserving the polarity induced by the B-quadric. It is not difficult to show that the groups of motions of the Euclidean R_3 and pseudo-Euclidean ${}^1R_3$ spaces are isomorphic to subgroups of motions of a biplanar space of parabolic type, preserving a B-polarity.

In connection with this there arises the question of classifying B-quadrics and establishing a correspondence between their types and the line geometries of three-dimensional spaces.

3. We shall carry out the classification of B-quadrics separately for each of the three types B_{n+1}.

A. **Hyperbolic type:** $\gamma^\alpha_\sigma \gamma^\sigma_\beta = \delta^\alpha_\beta$. The matrices of the absolute involution (γ^α_β), of the B-quadric $(a_{\alpha\beta})$, and of the B-motion (t^α_β) are reduced to the form:

$$(\gamma^\alpha_\beta) = \begin{pmatrix} -E_{m+1} & 0 \\ 0 & E_{n-m+1} \end{pmatrix}; \qquad (a_{\alpha\beta}) = \begin{pmatrix} -E_p & & & \\ & E_r & 0 & \\ & 0 & -E_q & \\ & & & E_s \end{pmatrix}; \qquad (t^\alpha_\beta) = \begin{pmatrix} P_{m+1} & 0 \\ 0 & Q_{n-m+1} \end{pmatrix},$$

where E_i is the identity matrix of order i; P_{m+1} and Q_{n-m+1} are arbitrary (square) matrices. The numbers p, q, r, s are related by the relations $p+r = m+1$, $q+s = n-m+1$. The index l ([3], p. 297) of the B-quadric is equal to $l = p+q$. If $m = n/2$ (n even), one can arrange that $p > q$.

The numbers p, q, and m form a complete system of invariants of the B-quadric. If $n = 2k,\ m = k$ (the proper biplanar space), one can introduce a canonical coordinate system in which

$$(\gamma^\alpha_\beta) = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}.$$

In this case the matrix of a B-quadric of zero signature can be reduced to the form

$$(a_{\alpha\beta}) = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}; \qquad C = \begin{pmatrix} E_p & 0 \\ 0 & E_{k-p+1} \end{pmatrix}. \tag{4}$$

Generally speaking, in this case, in the canonical coordinate system the matrix of a B-quadric cannot be reduced either to the form (4) or to diagonal form (the latter is possible when $p = q$).

B. **Elliptic type:** $\gamma^\alpha_\sigma \gamma^\sigma_\beta = -\delta^\alpha_\beta$. The matrices (γ^α_β), $(a_{\alpha\beta})$, and (t^α_β) are reduced to the form:

$$(\gamma^\alpha_\beta) = \begin{pmatrix} 0 & -E_{k+1} \\ E_{k+1} & 0 \end{pmatrix}; \qquad (a_{\alpha\beta}) = \begin{pmatrix} 0 & E_{k+1} \\ E_{k+1} & 0 \end{pmatrix}; \qquad (t^\alpha_\beta) = \begin{pmatrix} P_{k+1} & -Q_{k+1} \\ Q_{k+2} & P_{k+1} \end{pmatrix}$$

(cf. ([1], p. 93)).

C. **Parabolic type:** $\gamma^\sigma_\alpha \gamma^\beta_\sigma = 0$. The matrices (γ^α_β), $(a_{\alpha\beta})$, and (t^α_β) are reduced to the form

$$(\gamma^\alpha_\beta) = \begin{pmatrix} 0 & 0 & 0 \\ E_r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (a_{\alpha\beta}) = \begin{pmatrix} 0 & C_r & 0 \\ C_r & 0 & 0 \\ 0 & 0 & B_s \end{pmatrix}; \qquad (t^\alpha_\beta) = \begin{pmatrix} P_r & 0 & 0 \\ Q_r & P_r & T \\ S & 0 & R_s \end{pmatrix};$$

$$C_r = \begin{pmatrix} -E_k & 0 \\ 0 & E_{r-k} \end{pmatrix}; \qquad B_s = \begin{pmatrix} -E_l & 0 \\ 0 & E_{s-l} \end{pmatrix};$$

P_r, Q_r, R_s are square matrices, and S and T are arbitrary rectangular matrices.

A B-quadric of zero signature exists in B_{2n+1} in any of the indicated cases; moreover, if the matrix of the absolute collineation is changed in the corresponding manner, the matrix of the B-quadric will have the form

$$\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}.$$

The results obtained make it possible to carry out a classification of the subgroups of biplanar motions that leave invariant the polarity determined by the given B-quadric.

4. A. P. Shirokov ([6]) proved that the geometry of any symmetric conformally Euclidean space (an SC-space) can be realized as the internal geometry of a B-quadric normalized by means of an absolute involution.

Since the transformation group P_5 leaving invariant the Plücker hyperquadric Q_4 is isomorphic to the group of projective transformations in P_3, the classification of B-motions leaving Q_4 invariant makes it possible to distinguish all subgroups of projective transformations of three-dimensional space whose line geometries, under the mapping onto the Plücker hyperquadric, determine a symmetric space.

As a result we obtain Table 1. Cases 4–8 of this table correspond to the spaces considered by A. P. Norden ([2], p. 153). Clas-

Table 1

Type of envelope in P_5	No.	Additional data	Type of $^2SC_4$, dimension and characteristic in the completely geodesic case of reducibility	Type of three-dimensional space whose line geometry is realized in $^2SC_4$
Hyperbolic	1	$m = 0$	Of constant curvature	Symplectic
Hyperbolic	2	$m = 1p = 1$	Reducible, i.e., three-dimensional; the family, i.e., does not contain isotropic surfaces	Biaxial of elliptic type
Hyperbolic	3	$m = 1p = 2$	Reducible, i.e., three-dimensional; the family, i.e., contains 2 isotropic surfaces	Biaxial of hyperbolic type
Hyperbolic	4	$m = 2p = 2$	Reducible, i.e., two-dimensional hyperbolic spaces	Space with a line absolute
Hyperbolic	5	$m = 2p = 3$	Reducible, i.e., two-dimensional elliptic spaces	Elliptic
Elliptic	6	–	Irreducible	Hyperbolic
Parabolic	7	$r = 3k = 0$	Irreducible, type I	Euclidean
Parabolic	8	$r = 3k = 1$	Irreducible, type II	Pseudo-Euclidean

Type of envelope in P_5	No.	Additional data	Type of $^2SC_4$, dimension and characteristic in the completely geodesic case of reducibility	Type of three-dimensional space whose line geometry is realized in $^2SC_4$
Parabolic	9	$r = 2k = 0$	Irreducible, type III	Absolute, decomposes into 2 imaginary straight lines
Parabolic	10	$r = 2k = 1$	Irreducible, type IV	Absolute, decomposes into 2 real straight lines
Parabolic	11	$r = 1$	Of zero curvature	Absolute, 2 coincident straight lines

sification of SC-spaces agrees with the results of P. A. Shirokov ([7]). To the line geometry of all the spaces indicated in the last column of Table 1, the results of the work ([4]) are applicable. Using the projective interpretation of SC-spaces given by A. P. Shirokov ([6]), it is not difficult to construct a conformal interpretation of all these line geometries.

In this case the absolute invariant of two adjacent straight lines is determined by the quadratic form:

$$g_{ij} = \partial_i x \, \partial_j x \qquad (i, j = 1, \dots, 4),$$

where, as usual, $xy = a_{\alpha\beta} x^\alpha x^\beta$ $(\alpha, \beta = 1, \dots, 6)$, and the normalization of the points (straight lines) x is subject to the condition $xx = 1$ (see ([4]) and ([6]), p. 17).

Kuibyshev State
Pedagogical Institute

Received
21 IV 1963

CITED LITERATURE

[1] G. E. Izotov, *Izv. vyssh. uchebn. zaved., Matematika*, No. 1 (2), 89 (1958).
[2] A. P. Norden, *Izv. vyssh. uchebn. zaved., Matematika*, No. 4 (17), 145 (1960).
[3] B. A. Rozenfeld, *Non-Euclidean Geometries*, 1957.
[4] V. D. Tretyakov, *Volzhskii Mat. Sborn.*, No. 1 (1963).
[5] A. P. Shirokov, *Uch. zap. Kazansk. gos. univ.*, **114**, book 2 (1954).
[6] A. P. Shirokov, *Uch. zap. Kazansk. gos. univ.*, **116**, book 1, 15 (1956).
[7] P. A. Shirokov, *Izv. Kazansk. fiz.-matem. obshch.*, **11**, ser. 3, 9 (1938).

Note: Figure translations are in progress. See original paper for figures.