MATHEMATICS

B. V. FEDOSOV

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

B. V. FEDOSOV

ASYMPTOTIC FORMULAS FOR THE EIGENVALUES OF THE LAPLACE OPERATOR IN THE CASE OF A POLYGONAL DOMAIN

(Presented by Academician S. L. Sobolev on 13 February 1963)

We consider the boundary-value problem for the equation

$$\Delta u = -\nu^2 u$$

in a domain D with boundary condition $u|_\Gamma = 0$ or $\left.\frac{\partial u}{\partial n}\right|_\Gamma = 0$. The domain D is an arbitrary polygon (possibly nonconvex), and Γ is the polygonal line without self-intersections that bounds it.

Theorem. Let ν_n^2 be the n-th eigenvalue and

$$n(\nu) = \sum_{\nu_n \leqslant 1} 1.$$

Then

$$n(\nu) = \frac{S}{4\pi}\nu^2 + O(\nu), \tag{1}$$

$$\int_0^\nu n(t)\,dt = \frac{S}{12\pi}\nu^3 \mp \frac{l}{8\pi}\nu^2 + O(\nu), \tag{2}$$

where S is the area of D, and l is the perimeter of Γ. Here and below the minus sign corresponds to the boundary condition $u|_\Gamma = 0$, and the plus sign to the condition $\left.\frac{\partial u}{\partial n}\right|_\Gamma = 0$.

Formulas (1) and (2) are obtained from the expression for the trace of the Green function of the mixed boundary-value problem with the aid of V. A. Marchenko's Tauberian theorem for Fourier integrals*.

We begin by considering an auxiliary mixed boundary-value problem for the wave equation

$$\Delta w = \frac{\partial^2 w}{\partial t^2} \quad \text{in } D$$

with initial conditions

$$w|_{t=0} = 0, \qquad \left.\frac{\partial w}{\partial t}\right|_{t=0} = \delta\left(|\bar{x} - \bar{y}|\right); \qquad \bar{x}, \bar{y} \in D,$$

and boundary condition $w|_\Gamma = 0$ or $\frac{\partial w}{\partial n} = 0$. Here $|\bar{x} - \bar{y}|$ is the distance between the points $\bar{x}$ and $\bar{y}$.

Let D be a polygon with angles $\omega_n = \pi/\alpha_n$, $\alpha_n > 1/2$. Choose t_0 so small that the disk $K_{\bar{x},t_0}$ of radius t_0 centered at any point $\bar{x} \in D$ intersects no more than two adjacent sides of the polygon. For $t < t_0$, the function $w(\bar{x}, \bar{y}, t)$ can be found easily. Indeed, if $K_{\bar{y},t_0}$ lies entirely in D, then

$$w(\bar{x}, \bar{y}, t) = \frac{\sigma\left(t - |\bar{x} - \bar{y}|\right)}{2\pi\sqrt{t^2 - |\bar{x} - \bar{y}|^2}},$$

where

$$\sigma(t) = \begin{cases} 0, & \text{for } t \leqslant 0, \\ 1, & \text{for } t > 0; \end{cases}$$

if $K_{\bar{y},t_0}$ intersects only one side, then

$$w(\bar{x}, \bar{y}, t) = \frac{\sigma(t - |\bar{x} - \bar{y}|)}{2\pi\sqrt{t^2 - |\bar{x} - \bar{y}|^2}} \mp \frac{\sigma(t - |\bar{x} - \bar{y}'|)}{2\pi\sqrt{t^2 - |\bar{x} - \bar{y}'|^2}},$$

where $\bar{y}'$ is the point symmetric to $\bar{y}$ with respect to this side. The remaining points are near one of the corners, and during the time t_0 the influence of the other corners will not be felt. For such $\bar{y}$

$$w(\bar{x}, \bar{y}, t) = F(\varphi - \psi) \mp F(\varphi + \psi)$$

($\bar{x}$ and $\bar{y}$ have polar coordinates r, φ and s, ψ; here r and s are the distances from the vertex of the angle, and φ and ψ are measured from one of the sides of the angle). The function $F(\theta)$ is expressed in terms of a contour integral (see ([1]))*

---

* Such a plan was proposed by Prof. A. Ya. Povzner at I. M. Gelfand' s seminar at Moscow University in 19

Fig. 1

Figure 1: Fig. 1

$$F(\theta) = \frac{\alpha_n}{4\pi i} \int_{C_t} \frac{\operatorname{ctg} \alpha_n \frac{z-\theta}{2}\, dz}{2\pi\sqrt{t^2 - r^2 - s^2 + 2rs\cos z}}.$$

The contour C_t is determined as follows. In the z-plane draw vertical cuts from the branch points of the integrand to ∞ for $t > r + s$ (Fig. 1), and for $t < r + s$ draw cuts as shown in Fig. 2. Then the contour C_t consists of two lines joining the branch points (in the figures the contour C_t is indicated by a solid line). For $t < r + s$, $F(\theta)$ is equal to the sum of the residues at the poles of

$$\operatorname{ctg} \alpha_n \frac{z - \theta}{2}$$

inside C_t. For $t > r + s$, $F(\theta)$ is equal to the sum of the residues at the poles of

$$\operatorname{ctg} \alpha_n \frac{z - \theta}{2},$$

lying between $-\pi$ and π, plus the integral

$$\frac{\alpha_n}{4\pi i} \int_{S_t} \frac{\operatorname{ctg} \alpha_n \frac{z-\theta}{2}\, dz}{2\pi\sqrt{t^2 - r^2 - s^2 + 2rs\cos z}};$$

the contour S_t is indicated in Fig. 1 by a dashed line.

Fig. 1

Thus:

for $t < r + s$

$$w(\bar{x}, \bar{y}, t) = \sum_{|\varphi - \psi + 2\omega_n k| < \tau} \frac{1}{2\pi\sqrt{t^2 - r^2 - s^2 + 2rs\cos(\varphi - \psi + 2\omega_n k)}} \mp$$

$$\mp \sum_{|\varphi + \psi + 2\omega_n k| < \tau} \frac{1}{2\pi\sqrt{t^2 - r^2 - s^2 + 2rs\cos(\varphi + \psi + 2\omega_n k)}},$$

τ is the solution of the equation $t^2 - r^2 - s^2 + 2rs\cos\tau = 0$;

for $t > r + s$

$$w(\bar{x},\bar{y},t) = \sum_{|\varphi-\psi+2\omega_n k|<\pi} \frac{1}{2\pi\sqrt{t^2-r^2-s^2+2rs\cos(\varphi-\psi+2\omega_n k)}} \mp$$

$$\mp \sum_{|\varphi+\psi+2\omega_n k|<\pi} \frac{1}{2\pi\sqrt{t^2-r^2-s^2+2rs\cos(\varphi+\psi+2\omega_n k)}} +$$

$$+\frac{\alpha_n}{2\pi i}\int_{S_t} \frac{\operatorname{ctg}\alpha_n \frac{z-\varphi+\psi}{2}\,dz}{2\pi\sqrt{t^2-s^2-r^2+2rs\cos z}} \mp \frac{\alpha_n}{4\pi i}\int_{S_t} \frac{\operatorname{ctg}\alpha_n \frac{z-\varphi-\psi}{2}\,dz}{2\pi\sqrt{t^2-r^2-s^2+2rs\cos z}};$$

k is an integer.

---

[*] In (1) the Green' s function of the equation $\Delta u = k^2 u$ is constructed for an angle, but with inessential changes these arguments are also applicable to the mixed problem.

Putting $\bar{x} = \bar{y}$ and integrating over the domain, we obtain

$$w(t) = \int_D w(\bar{x},\bar{y},t)\,d\bar{x} = \frac{S}{2\pi t} \mp \frac{l}{8} + ct \quad \text{for } t < t_0;$$

$c = \sum c_n$, where c_n are constants depending on the magnitudes of the angles ω_n.

Starting from the Fourier method, we can obtain the following expression for $w(\bar{x},\bar{y},t)$:

$$w(\bar{x},\bar{y};t) = \sum_{n=1}^{\infty} \frac{u_n(\bar{x})u_n(\bar{y})}{\nu_n} \sin\nu_n t;$$

here $u_n(\bar{y})$ is the normalized eigenfunction of the problem corresponding to the eigenvalue ν_n^2.

For $w(t)$ we hence obtain

$$w(t) = \sum_{n=1}^{\infty} \frac{\sin\nu_n t}{\nu_n} = \int_0^{\infty} \frac{\sin\nu t}{\nu}\,dn(\nu).$$

The convergence of the series and the integral is understood in the "weak" sense on three-times differentiable finite functions.

Thus,

Fig. 2

Figure 2: Fig. 2

$$\int_0^\infty \frac{\sin \nu t}{\nu}\, dn(\nu) = \frac{S}{2\pi t} \mp \frac{l}{8} + ct \quad \text{for } t < t_0. \tag{3}$$

The asymptotic formulas for $n(\nu)$ are obtained from formula (3) with the help of the following Tauberian theorem for Fourier integrals (see ([2,3])).

Let $\rho(\nu)$ satisfy the following conditions:

a) on every finite interval the function $\rho(\nu)$ has bounded variation;

b) as $\mu \to +\infty$,

$$\bigvee_{\mu}^{\mu+1} \rho(\nu) = O(\mu^p), \qquad p > 0;$$

c) for every even function $g_\Lambda(t)$ having a bounded $(p+2)$-nd derivative and equal to zero outside the interval $(-\Lambda, \Lambda)$,

$$\int_0^\infty h(\nu)\, d\rho(\nu) = 0,$$

where

$$h(\nu) = \frac{1}{\pi} \int_0^\Lambda g_\Lambda(t) \cos \nu t \, dt.$$

Fig. 2

Then, as $\mu \to +\infty$, the estimate

$$\int_0^\mu \left(1 - \frac{\nu^2}{\mu^2}\right)^r d\rho(\nu) = O(\mu^{p-r}), \qquad r \geq 0.$$

is valid.

By the method expounded, for example, in ([4]), one can verify that

$$\bigvee_{\mu}^{\mu+a} n(\nu) < A\mu,$$

where a is fixed.

Put

$$n_1(\nu) = n(\nu) - \frac{S}{4\pi}\nu^2 \pm \frac{l}{4\pi}\nu - c; \qquad n_1(0) = 0.$$

Then

$$\int_0^\infty \frac{\sin \nu t}{\nu}\, dn_1(\nu) = 0 \quad \text{for } t < t_0.$$

Take an even, three-times differentiable function $f(t)$, equal to zero outside the interval $(-t_0, t_0)$, and the function $h(\nu)$, equal to

$$h(\nu) = \frac{1}{\pi}\int_0^{t_0} f(t)\cos \nu t\, dt = -\frac{1}{\pi\nu}\int_0^{t_0} f'(t)\sin \nu t\, dt.$$

Then

$$\int_0^\infty h(\nu)\, dn_1(\nu) = \int_0^\infty \frac{1}{\pi\nu}\left(\int_0^{t_0} f'(t)\sin \nu t\, dt\right) dn_1(\nu) =$$

$$= -\frac{1}{\pi}\int_0^{t_0} f'(t)\left(\int_0^\infty \frac{\sin \nu t}{\nu}\, dn_1(\nu)\right) dt = 0.$$

It is clear that

$$\bigvee_{\mu}^{\mu+a} n_1(\nu) = O(\mu).$$

Consequently, by Tauber’ s theorem,

$$\int_0^\mu \left(1 - \frac{\nu^2}{\mu^2}\right)^r dn_1(\nu) = O(\mu^{1-r}), \qquad r \geqslant 0.$$

For $r = 0$ we obtain formula (1). For $r = 1$ we have:

$$\int_0^\mu \left(1 - \frac{\nu^2}{\mu^2}\right) dn_1(\nu) = O(1).$$

Integrating twice by parts, we obtain

$$\frac{N_1(\mu)}{\mu} - \frac{1}{\mu^2}\int_0^\mu N_1(\nu)\,d\nu = O(1); \qquad N_1(\mu) = \int_0^\mu n_1(\nu)\,d\nu,$$

i.e.

$$\left(\frac{1}{\mu}\int_0^\mu N_1(\nu)\,d\nu\right)' = O(1).$$

Hence

$$\int_0^\mu N_1(\nu)\,d\nu = O(\mu^2),$$

and for $N_1(\mu)$ we have $N_1(\mu) = O(\mu)$, which gives formula (2).

From formulas (1) and (2) one can also obtain the formula

$$\int_0^\mu (\mu-\nu)^r\,dn_1(\nu) = O(\mu) \qquad \text{for } 0 \leqslant r \leqslant 1,$$

i.e.

$$\int_0^\mu (\mu-\nu)^r\,dn(\nu) = \frac{S}{2\pi(r+1)(r+2)}\mu^{r+2} \mp \frac{l}{4\pi(r+1)}\mu^{r+1} + O(\mu).$$

The author takes this opportunity to express his deep gratitude to Prof. V. B. Lidskii for posing the problem and for constant assistance in the work.

Moscow Institute of Physics and Technology

Received
2 II 1963

REFERENCES

1. G. Herglotz, *Math. Ann.*, **124**, 219 (1952).
2. V. A. Marchenko, *Izv. AN SSSR, Ser. Mat.*, **19**, 381 (1955).
3. B. M. Levitan, Appendix V to the book by E. C. Titchmarsh, *Eigenfunction Expansions*, Part I, 1960.
4. B. M. Levitan, Appendix VI to the book by E. C. Titchmarsh, *Eigenfunction Expansions*, Part II, 1961.