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Abstract

Full Text

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CATEGORIES WITH DIVISIBILITY AND INTEGRAL REPRESENTATIONS

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1. In this section we shall for the most part adhere to the notation adopted in ([1]). For an arbitrary category K one can construct a category $\widetilde{K}$, whose objects are the objects of K, and $\widetilde{H}(a,b)$ is the set of all nonempty subsets of $H(a,b)$; moreover, if $S \in \widetilde{H}(a,b)$, $T \in \widetilde{H}(b,c)$, then ST will be understood to be the collection of all mappings of the form $\sigma\tau$, where $\sigma \in S$, $\tau \in T$. We shall regard the category K as a subcategory of the category $\widetilde{K}$, identifying a subset consisting of a single mapping with this mapping. We shall call a pair (K,L) a **category with divisibility** if L is a subcategory of the category $\widetilde{K}$ containing all objects (i.e., all identity mappings) from K.

Denote by $L(a,b)$ the set of mappings from a to b in the category L ($L(a,b) \subseteq \widetilde{H}(a,b)$). We shall say that a divides b (a/b) if $L(a,b)$ is nonempty. The divisibility relation is, obviously, reflexive and transitive, i.e., it is a quasi-ordering relation ([2]) on the class of objects of the category K. We shall say that the objects a and b are associated if a/b and b/a. The association relation is an equivalence relation, and the divisibility relation induces a partial ordering relation on the collection of classes of associated objects.

In what follows we shall assume that the category K has zero objects and, consequently, zero mappings. We shall say that (K,L) is a category with nonzero divisibility if, for any two nonzero objects a and b of the category K, the zero mapping ω_{ab} is not contained in $L(a,b)$.

Proposition 1. *If (K,L) is a category with nonzero divisibility, a and b are associated nonzero objects of K, and the semigroup $H(a,a)$ is finite, then there exist $\varphi : a \to b$, $\psi : b \to a$ such that $\varphi\psi$ is a nonzero idempotent in $H(a,a)$, and $\psi\varphi$ is a nonzero idempotent in $H(b,b)$.*

Indeed, the objects a and b are associated, hence $u \in L(a,b)$, $v \in L(b,a)$, $uv \in W$. W is a subsemigroup of the semigroup $H(a,a)$. Obviously, for every n, $W^n \neq 0$. But then, as is known ([3]), there is an $\alpha \in W$ such that, for every n, $\alpha^n \neq 0$. Since in a finite semigroup every element in some power is equal to an idempotent, it follows that $\alpha^n = \beta$, $\beta^2 = \beta \neq 0$, $\beta = \varphi\psi$, where $\varphi \in H(a,b)$, $\psi \in H(b,a)$. Putting $\psi = \psi\varphi\psi$, we obtain $\varphi\psi = \beta \in H(a,a)$, $\psi\varphi = \gamma \in H(b,b)$, $\gamma^2 = \gamma \neq 0$.

Proposition 1 can be somewhat strengthened by replacing the requirement of finiteness of $H(a,a)$ by certain weaker conditions. In particular, if K is an additive category, then it suffices to require that the ring $H(a, a)$ satisfy the maximality and minimality conditions. On the other hand, it is easy to show that Proposition 1 ceases to be true if no finiteness-type restrictions are imposed on $H(a, a)$.

2. Let now K be a category of modules over an associative ring with identity. Let $A, B \in K$, A' be a submodule of the module A, and T be a subset

in $\mathrm{Hom}(A, B)$; denote by $A'T$ the submodule of the module B generated by elements of the form $a't$, where $a' \in A'$, $t \in T$. We shall call the set T epimorphic if $AT = B$.* Since the product of two epimorphic sets is again an epimorphic set, and a set consisting of a single identity mapping is also epimorphic, we can construct L, denoting by $L(A, B)$ the set of epimorphic sets lying in $\mathrm{Hom}(A, B)$, i.e., introduce a divisibility relation on the category of K-modules. Now A/B means that $A \,\mathrm{Hom}(A, B) = B$.

A decomposition of a module A into a direct sum $A_1 \oplus \cdots \oplus A_k$ will be called normal if A_i divides A_j for $i < j$. A module that cannot be decomposed into a normal direct sum will be called normally indecomposable.

Proposition 2. *Let M be a commutative Noetherian ring with identity, and let U be an ideal of the ring M such that*

$$\bigcap_{k=1}^{\infty} U^k = 0,$$

the factor ring M/U satisfies the minimum condition, and the ring M is complete as a topological space with the topology induced by the ideals U^k. Let K be the category of finitely generated Λ-modules, where Λ is an M-algebra with a finite number of generators. Finally, let A and B be modules from K, with B dividing A and B normally indecomposable. Then every exact sequence $A \to B \to 0$ splits.

Let us note that if M is a ring satisfying the minimum condition, then the zero ideal may be taken as U. Let us also note that the assertion will hold all the more if B is indecomposable in the usual sense. The proof of Proposition 2 in the case where M is a ring satisfying the minimum condition and the module B is indecomposable coincides exactly with the proof of Proposition 1. In the general case the proof becomes only technically more complicated.

3. Let K be an arbitrary category. A subobject (b, μ) of an object a will be called a supercharacteristic subobject if every $\nu : b \to a$ can be represented in the form $\alpha\mu$, where $\alpha : b \to b$.

In group theory, as is well known, characteristic and fully characteristic subgroups are considered. A supercharacteristic subgroup, i.e. a subgroup that is a supercharacteristic subobject in the sense of the definition given above, is, of course, fully characteristic and, a fortiori, characteristic. However, every proper subgroup of an infinite cyclic group is fully characteristic, but not a supercharacteristic subgroup.

We shall need the concept of a supercharacteristic submodule, for which one may also give the following definition: a submodule A' of a module A is called supercharacteristic if $A' \operatorname{Hom}(A', A) = A'$. The following almost obvious proposition establishes the connection between supercharacteristic submodules and the divisibility relation on the category of modules introduced in the preceding point.

Proposition 3. *The module $A \operatorname{Hom}(A, B)$ is a supercharacteristic submodule of the module B for every A. The mapping that assigns to each module A of the category K the supercharacteristic submodule $A \operatorname{Hom}(A, B)$ of the module B induces a homomorphism of the partially ordered set of classes of associated modules of the category m into the set of supercharacteristic submodules of the module B, partially ordered by inclusion.*

From Proposition 2 it follows immediately that

Proposition 4. *If K is a category of modules satisfying the conditions of Proposition 2, then the exact sequence $A \xrightarrow{\varphi} B \to 0$, where*

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* Let us note that the notion of an epimorphic, and also of a monomorphic, set of mappings can be naturally defined in an arbitrary category, by analogy with the usual definitions of epimorphism and monomorphism in category theory.

the module B is normally indecomposable, splits if and only if $A'\varphi = B$, where $A' = B \operatorname{Hom}(B, A)$.

4. In this item Z is the ring of rational integers, Z_p is the ring of integral p-adic numbers, Λ is a finitely generated Z-free Z-algebra with identity, and Λ_p is a finitely generated Z_p-free Z_p-algebra with identity. We shall call the category of finitely generated Z-(Z_p-)free Λ-(Λ_p-)modules the category of integral (p-adic) representations.

First of all, note that in the category of p-adic representations the conditions, and hence also the assertions, of Propositions 2 and 4 are satisfied. From Proposition 2 and the uniqueness of decomposition into indecomposables in the category of p-adic representations ([4]) it follows:

Proposition 5. *In each class of associated modules of the category of p-adic representations there is one and only one normally indecomposable module.*

If A is a module lying in the category of integral representations, then for every p the module $A_p = A \otimes_Z Z_p$ is defined, lying in the category of p-adic representations of the ring $\Lambda_p = \Lambda \otimes_Z Z_p$.

Proposition 6. *A/B if and only if A_p/B_p for all p.*

The author does not know whether, in the category of integral representations, the assertions of Propositions 2 and 4 are satisfied. However, it can be proved that

Proposition 7. *Let K be a category of modules satisfying the conditions of Proposition 2, or the category of integral representations. Then the exact sequence $A \xrightarrow{\varphi} B \to 0$ splits if and only if, for every supercharacteristic submodule B' of the module B such that $B' \operatorname{Hom}(B, B') = B'$, the equality $A'\varphi = B'$ holds, where $A' = B' \operatorname{Hom}(B, A)$.*

Note that if Λ is a semisimple ring, then, since there exists only a finite number of nonisomorphic modules of a given dimension ([5]), every module from the category of integral representations of the ring Λ has only a finite number of distinct supercharacteristic submodules.

From Proposition 7 one may obtain the following corollary: If A and B are two modules from the category K of integral representations, and A_p is isomorphic to B_p for every p, then for some n $A^{(n)} = B \oplus X$ and $B^{(n)} = A \oplus Y$, where $A^{(n)}$ ($B^{(n)}$) is the direct sum of n copies of the module A (B), and X and Y are certain modules from K.

Let us also note that everything said about integral and p-adic representations carries over almost verbatim to representations over Dedekind rings and their completions.

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Note: Figure translations are in progress. See original paper for figures.