# D. P. Milman, V. D. Milman

1963

SovietRxiv

## Abstract

## Full Text

# D. P. Milman, V. D. Milman

## SOME GEOMETRIC PROPERTIES OF NONREFLEXIVE SPACES

*(Presented by Academician S. L. Sobolev on 21 III 1963)*

**1.** Two convex cones $K' \subset B_1$ and $K'' \subset B_2$, where $B_1$ and $B_2$ are Banach spaces, will be called **locally isomorphic** if between them one can establish a one-to-one linear correspondence $\varphi$: $\varphi(K') = K''$, under which the relations

$$\lim_{n\to\infty} \|x'_n - x'_0\|_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\varphi(x'_n) - x''_0\|_2 = 0$$

hold only jointly, and, moreover, if $x'_0 \in K'$, then $x''_0 = \varphi(x'_0) \in K''$, and conversely.

Let us note that from the local isomorphism of $K'$ and $K''$ it does not follow that the relations

$$\lim_{n\to\infty} \|x'_n - y'_n\|_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\varphi(x'_n) - \varphi(y'_n)\|_2 = 0$$

hold only jointly. Indeed, if the latter relations hold for arbitrary sequences $x'_n, y'_n \in K'$, then, as is easy to see, the closed linear hulls of the cones $K'$ and $K''$ are isomorphic as Banach spaces; however, a local isomorphism of cones, as will be seen from what follows, does not imply, in general, isomorphism of their closed linear hulls.

In what follows, $\{e_n\}_{1\le n<\infty}$ denotes the natural basis in the space $l_1$ (of absolutely convergent sequences of numbers), and $K_l$ is the smallest convex closed cone containing $\{e_n\}_{1\le n<\infty}$.

**Theorem 1.** 1) In order that a Banach space $B$ be nonreflexive, it is necessary and sufficient that it contain a cone $K$ locally isomorphic to $K_l$.

2) The indicated cone $K$ can be chosen so that, in addition, the sequence $\{\varphi(e_n)\}_{1\le n<\infty} \subset B$, where $\varphi$ denotes the correspondence establishing the local isomorphism $\varphi(K_l) = K$, has a biorthogonal sequence of linear functionals.

The sequence $\{x_n\}_{1\le n<\infty}$, $x_n = \varphi(e_n)$, will be called below the $l$-basis of the cone $K$.

**Remark to Theorem 1.** Using Theorem 2 of A. Pełczyński' s paper ([3]), one can prove that the cone $K$ in part 2) of our theorem 1 can be chosen so that the $l$-basis $\{x_n\}$ in it is a basic sequence (i.e., a basis in its closed linear hull).

For an illustration of Theorem 1, let us give an example of a cone $K$, locally isomorphic to $K_l$, in the space $c_0$ (of sequences of numbers converging to zero). It suffices to specify an $l$-basis of such a cone. It is the sequence $\{x_n\}_{1\leq n<\infty}$, where $x_n$ has its first $n$ coordinates equal to 1, and the remaining ones equal to zero. One can show that the sequence $\{x_n\}_{1\leq n<\infty}$ in this example is also a conditional basis in the space $c_0$.

Relying on Theorem 1, one can establish the following result:

**Theorem 2.** For the reflexivity of a Banach space $B$, it is necessary and sufficient that every affine continuous mapping of an arbitrary nonempty convex closed bounded set onto itself have a fixed point.

**2.** The results of the preceding section use the Šmulian-Eberlein theorem ([1,2]), according to which a necessary and sufficient condition

of nonreflexivity of a Banach space $B$ is the existence in it of a countable system of nonempty convex closed bounded sets $\{G_n\}_{1<n<\infty}$, $G_{n+1}\subset G_n$, having empty intersection. In what follows we shall call such a system a **deposit with empty intersection** and write $\pi=\{G_n\}$. By studying such deposits one discovers a number of geometric properties of nonreflexive Banach spaces, some of which are set out below.

We shall call a deposit $\pi_1=\{G_n^1\}$ **subordinate** to a deposit $\pi=\{G_n\}$ and write $\pi_1\prec\pi$, if $G_n^1\subset G_n$ for all $n$, $1\leq n<\infty$.

Denote by $r(x,G_n)$ the distance from the point $x$ to the set $G_n$; $r(x,\pi)=\lim_{n\to\infty} r(x,G_n)$; $r(\pi)=\lim_{n\to\infty}\inf_{x\in G_n} r(x,\pi)$*. One can show that the number $r(\pi)$ is the exact lower bound of the numbers

$$\lim_{n\to\infty}\lim_{m\to\infty}\|x^n-y^{n+m}\|,$$

where $\{x^n,y^n\in G_n\}$. It is obvious that for a deposit $\pi_1\prec\pi$ one has $r(\pi_1)\geq r(\pi)$.

A deposit $\pi=\{G_n\}$ with empty intersection will be called **$\omega$-split** if $r(\pi)>0$.

A bounded sequence of elements $\{u_n\}_{1<n<\infty}$ of the space $B$ will be called **$\omega$-split** if there exists $\delta>0$ such that $\delta\leq\lim_{m\to\infty}\|u_{mn_m}-u_{n_m\omega}\|$, where $u_{mn}$ and $u_{n\omega}$ denote arbitrary elements from the convex hulls of the elements $\{u_j\}_{m<j\leq n}$ and $\{u_j\}_{n<j<\infty}$, respectively, and $n_m$ is an arbitrary number greater than $m$. The exact upper bound of such numbers $\delta$, taken over all subsequences $\{u_{n_k}\}_{1\leq k<\infty}$, will be called the **index of $\omega$-splitness** of the sequence $\{u_n\}_{1\leq n<\infty}$.

**Theorem 3.** 1) *For every deposit $\pi$ with empty intersection there is a subordinate deposit $\pi_0\prec\pi$ which is $\omega$-split; every nonreflexive space contains an $\omega$-split deposit.* 2) *Let the deposit $\pi_0=\{G_n^0\}$ be $\omega$-split; then from every sequence $\{x^n\in G_n^0\}_{1\leq n<\infty}$ one can extract a subsequence $\{u_n\}_{1\leq n<\infty}$ which is $\omega$-split.* 3)

*In order that the space $B$ be nonreflexive it is necessary and sufficient that it contain an $\omega$-split sequence.*

**Remark.** Let $\pi = \{G_n\}$. The number $r(\pi)$ is equal to the exact lower bound of the indices of $\omega$-splitness over all sequences $\{x_n\}$, $x_n \in G_n$.

3. Denote by $d(M)$ the diameter of the set $M \subset B$. The number $d(\pi) = \lim_{n\to\infty} d(G_n)$ will be called the **diameter** of the deposit $\pi = \{G_n\}$. It is obvious that if $\pi_1 \prec \pi$, then $d(\pi_1) \leq d(\pi)$.

A deposit $\pi = \{G_n\}$ will be called $\omega$**-diametral** if for any elements $y_n, x_n \in G_n$ one has

$$\lim_{n\to\infty} \lim_{m\to\infty} \|x_n - y_{n+m}\| = d(\pi) > 0.$$

A bounded sequence of elements $\{u_n\}_{1\leq n<\infty}$ of the space $B$ will be called $\omega$**-diametral** if for any elements $u_{mn}$ from the convex hull of $\{u_j\}_{m<j\leq n}$ and $u_{n\omega}$ from the convex hull of $\{u_j\}_{n<j<\infty}$ one has $\lim_{m\to\infty} \|u_{mn_m} - u_{n_m\omega}\| = \lim_{n\to\infty} d(\{u_j\}_{n<j<\infty})$, where $n_m$ is an arbitrary

* One can give an example of a deposit for which

$$\lim_{n\to\infty} \inf_{x\in G_n} r(x,\pi) \neq \inf_{x\in G_1} r(x,\pi).$$

number greater than $m$, and $d(\{u_j\}_{n<j})$ denotes the diameter of the sequence $\{u_j\}_{n<j<\infty}$.

It can be proved that a nesting $\pi = \{G_n\}$ is $\omega$-diametral if and only if two conditions are simultaneously satisfied: a) $d(\pi) = r(\pi)$; b) for any point $x \in G_1$ one has

$$r(x,\pi) = \lim_{n\to\infty} r(x, y_n)$$

for an arbitrary sequence $\{y_n\}$, $y_n \in G_n$, $1 \leq n < \infty$,–the property of "compactness of the nesting" with respect to any point of the set $G_1$.

**Theorem 4.** 1) *For every nesting $\pi$ with empty intersection there is an $\omega$-diametral nesting $\pi_0 < \pi$ subordinate to it; every nonreflexive space contains an $\omega$-diametral nesting.* 2) *Let the nesting $\pi_0 = \{G_n\}$ be $\omega$-diametral; then from any sequence $\{x_n\}_{1\leq n<\infty}$, $x_n \in G_n$, one can choose a subsequence that is $\omega$-diametral.* 3) *Every nonreflexive Banach space contains a cone $K$, locally isomorphic to the cone $K_l$, and moreover such that its $l$-basis is an $\omega$-diametral sequence and a basis in its closed linear span.*

An example of an $\omega$-diametral sequence is the natural basis $\{e_n\}_{1\leq n<\infty}$ of the space $l_1$. In the space $c_0$ (sequences of numbers converging to zero) an example of an $\omega$-diametral sequence is the sequence $\{x_n\}_{1\leq n<\infty}$, where $x_n$ has its first $n$ coordinates equal to 1, and the remaining ones equal to zero.

A consequence of Theorem 4 is the result formulated below, which is directly connected with the fact [4] that every uniformly convex Banach space is reflexive.

We shall call an $n$-dimensional simplex, one of whose vertices is the origin, $\varepsilon$-directionally normalized $(1 > \varepsilon \geq 0)$ if there exists a numbering of its vertices $\{z_k\}_{0\leq k\leq n}$, $z_0 = 0$, such that for any $j$, $0 \leq n < j$, one has

$$1 \geq \|x_j - y_j\| \geq 1 - \varepsilon,$$

where $x_j$ is any element of the convex hull of $\{z_k\}_{0\leq k\leq j}$, and $y_j$ is any element of the convex hull of $\{z_k\}_{j<k\leq n}$.

**Corollary to Theorem 4.** *If the space $B$ is nonreflexive, then for every $\varepsilon > 0$ it contains an $\varepsilon$-directionally normalized simplex of arbitrarily high dimension.*

Odessa Electrotechnical Institute of Communications
Physico-Technical Institute of Low Temperatures
of the Academy of Sciences of the Ukrainian SSR

Received
14 III 1963

## References

1. V. L. Shmulian, Mat. sborn., **5** (47), 317 (1939).
2. W. A. Eberlein, Proc. Nat. Acad. Sci. USA, **33**, 51 (1947).
3. A. Pełczyński, Studia Math., **21**, 371 (1962).
4. D. P. Milman, DAN, **20**, 243 (1938).

*Note: Figure translations are in progress. See original paper for figures.*