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**Abstract**

**Full Text**

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## **Bicompact Sets and the Topology of Spaces**

*(Presented by Academician P. S. Aleksandrov on 20 XI 1963)*

### **I.**

In what follows, all spaces are assumed to be Hausdorff. Among the subsets of these spaces, the greatest interest is undoubtedly presented by bicomacts—these sets are invariant under embeddings into other spaces and in this respect resemble points; on the other hand, knowledge of all bicomact subsets of a space very often (for example, in the case of metric spaces) uniquely determines the topology of this space.

Let us note that the system of all closed sets does not have the first property, while the system of all one-point sets does not have the second. It is therefore natural to study and classify topological spaces by imposing various restrictions on the system of bicomact sets. We shall begin with metrization criteria obtained in this way.

**Definition 1.** A base  $B$  of a space  $X$  will be called **strongly uniform** if, for every bicomact set  $\Phi \subseteq X$  and every open set  $U \supseteq \Phi$ , the set of elements of  $B$  that meet both  $\Phi$  and  $X \setminus U$  is empty or finite.

This is a strengthening of the notion of a uniform base introduced by P. S. Aleksandrov in <sup>(1)</sup>.

**Theorem 1.** *In order that a space be metrizable, it is necessary and sufficient that it have a strongly uniform base.*

The notion of a refining sequence of covers is well known; it was first introduced into mathematics by P. S. Aleksandrov. If, following the basic idea of our note, in this definition we replace the word “point” by the words “bicomact set,” we arrive at the following definition.

**Definition 2.** A sequence of open covers  $\varphi = \{\gamma_n\}$  of a space  $X$  is called **strongly refining** if, for every bicomact set  $\Phi \subseteq X$  and every neighborhood  $U$  of it, there is a cover  $\gamma_{n_0} \in \varphi$  such that  $\gamma_{n_0}(\Phi)$ —the body of the star of the cover  $\gamma_{n_0}$  with respect to the set  $\Phi$ —is contained in  $U$ .

The class of spaces possessing countable refining sequences of covers is much broader than the class of metric spaces (these spaces have been, and are, widely studied; under certain additional restrictions they are called Moore spaces). At the same time, the following holds:

**Theorem 2.** *The class of spaces possessing countable strongly refining sequences of covers coincides exactly with the class of metric spaces.*

Bicompact sets can also be usefully employed in formulating conditions sufficient for preserving metrizability of spaces under mappings.\*

Let us first recall that a mapping  $f : X \rightarrow Y$  is called **quotient-open** if a set  $G \subseteq Y$  is open if and only if its full inverse image in  $X$ , i.e. the set  $f^{-1}G$ , is open.

Our interest in bicompact sets compels us to draw attention to the following class of mappings, previously considered by Bourbaki under the name of proper mappings.

\* In this note we consider only continuous mappings.

**Definition 3.** A mapping  $f : X \rightarrow Y$  is called a  $k$ -mapping if, for every bicompatum  $\Phi \subseteq Y$ , the set  $f^{-1}\Phi \subseteq X$  is also a bicompatum. As is known, the class of  $k$ -mappings includes all perfect, i.e. closed bicompat, mappings.

**Theorem 3\*.** *The image of a metrizable space under a quasi-open  $k$ -mapping is metrizable.*

Theorem 3 is quite general, since both open and closed mappings are quasi-open.

The proof of Theorem 3 is based on the notion of a  $k$ -space, whose topology is completely determined by the family of its bicompat subsets according to the following rule: a set is closed in it if and only if its intersection with any bicompat subset of the space is bicompat (or, what is the same, closed!). The  $k$ -spaces include all locally bicompat spaces and all spaces satisfying the first axiom of countability (see, for example, (3)).

II. In this section we shall present results concerning the connection between topological spaces and their mappings and  $k$ -spaces and  $k$ -mappings, and shall also report new information on the properties of  $k$ -spaces.

§ 1. Alongside  $k$ -mappings it is natural to single out their special case— $k$ -extensions, i.e. extensions in the usual sense of one topological space onto another which are at the same time  $k$ -mappings. In other words, a mapping  $f : X \rightarrow Y$  is a  $k$ -extension if and only if it is one-to-one, continuous (in one direction), and on each bicompat set of either of the spaces  $X$  or  $Y$  is a homeomorphism.

First of all,  $k$ -extensions make it possible to attach arbitrary topological spaces to  $k$ -spaces by the following obvious theorem\*\*.

**Theorem 4.** *Every topological space  $X$  is a  $k$ -extension of one and only one  $k$ -space.*

We shall denote this  $k$ -space by  $\tilde{X}$  and call it the  $k$ -image of the space  $X$ .

From Theorem 4 follows the following characterization of  $k$ -spaces.

**Corollary 1.** *A space is a  $k$ -space if and only if it is not a  $k$ -extension of any other space.*

Theorem 4 means that the totality of all topological spaces splits into pairwise disjoint classes in such a way that each class contains a unique  $k$ -space, and this  $k$ -space is maximal among the spaces of this class in the sense that each of them is its  $k$ -extension. (In connection with this fact many questions arise.) However, the connection between general topological spaces and  $k$ -spaces goes still deeper than what Theorem 4 establishes.

**Theorem 5.** *Let  $f : X \rightarrow Y$  be a continuous mapping. Then the mapping  $f : \tilde{X} \rightarrow \tilde{Y}$  is also continuous. (The converse is false.)*

To avoid confusion, we shall denote the mapping  $f : \tilde{X} \rightarrow \tilde{Y}$  by  $\tilde{f}$ .

**Corollary 1.** *If  $f$  is a  $k$ -mapping, then  $\tilde{f}$  is a perfect mapping.*

§ 2.1. **Theorem 6.** *The sum of a finite number of  $k$ -spaces that are closed (in the sum) is a  $k$ -space.*

In this assertion one cannot omit the word “closed,” or speak of a countable number of summands.

\* This is a generalization of a theorem of V. I. Ponomarev <sup>(2)</sup>: a paracompact space that is an open bicomact image of a metrizable space is metrizable.

\*\* Cohen <sup>(5)</sup> was apparently the first to note this fact, but he formulated it in the form of a considerably weaker theorem: every space is a  $k$ -extension of a  $k$ -space.

2. A closed subset of a  $k$ -space is again a  $k$ -space.

It is less obvious that

3. An open subset of a regular  $k$ -space is a  $k$ -space.

This property will follow from the following theorem:

4. **Theorem 7.** *A space  $X$  that is locally a  $k$ -space \* is a  $k$ -space.*
5. Under a  $k$ -mapping onto a  $k$ -space, weight cannot increase. This assertion may be regarded as a generalization of a well-known theorem of P. S. Aleksandrov.
6. **Theorem 8.** *Let  $f : X \rightarrow Y$  be a  $k$ -mapping of a space  $X$  onto a  $k$ -space  $Y$ . Then  $X$  is a  $k$ -space.*

As Cohen proved, when passing to the image the property of a space of being a  $k$ -space is preserved for a much broader class of mappings, namely for all quasi-open mappings.

7. The connection between continuous mappings and  $k$ -mappings is deepened by the following simple theorem:

**Theorem 9.** *Let  $f$  be an arbitrary continuous mapping of a completely regular topological space  $X$  onto a regular space  $Y$ . Then there exists an extension  $\tilde{X} \supseteq X$  of the space  $X$  to which the mapping  $f$  extends in such a way that the extended mapping  $\tilde{f} : \tilde{X} \rightarrow Y$  turns out to be a closed  $k$ -mapping. If  $Y$  is a  $k$ -space, then  $\tilde{X}$  is a  $k$ -extension of the space  $X$ , and in that case there exists no larger extension  $\hat{X}$  of the space  $X$ :  $\tilde{X} \supset \hat{X} \supseteq X$ , to which the mapping  $f$  would extend as a  $k$ -mapping.*

8. Finally, we give a general theorem relating  $k$ -mappings and perfect mappings.

**Theorem 10.** *Every continuous  $k$ -mapping of a  $k$ -space  $X$  decomposes into the superposition of a closed  $k$ -mapping (i.e. a perfect mapping) onto a  $k$ -preimage of the space  $Y$  and a  $k$ -condensation.*

Here it is appropriate to recall that, by a result of V. I. Ponomarev, a perfect mapping in turn decomposes into the superposition of a monotone and a zero-dimensional mapping.

**Corollary.** *If a space  $X$  is the image of a metrizable space under a  $k$ -mapping, then the  $k$ -preimage of the space  $X$  is metrizable.*

9. **Theorem 11.** *A normal  $k$ -space  $X$  is paracompact if and only if in each of its open covers  $\xi$  one can inscribe such an open cover  $\eta$  that every bicomact  $\Phi \subseteq X$  meets only a finite number of its elements.*

10. Let us note one intricate property of  $k$ -spaces: if  $Q$  is some non-closed subset of a  $k$ -space  $X$ , then there exist a bicomact set  $\Phi \subseteq X$  and a point  $q \in \text{fr } Q$  such that

$$q \in \text{fr } \Phi \cap Q = [\Phi \cap Q] \setminus (\Phi \cap Q).$$

This condition is necessary and sufficient for the space  $X$  to be a  $k$ -space. In the case of spaces with the first axiom of countability, it means that the point can be approached simply by a countable sequence of points converging to it, all elements of which belong to  $Q$ .

From this we can in particular see that spaces with the first axiom of countability are  $k$ -spaces.

However, here one can note a class of spaces lying between spaces with the first axiom of countability and  $k$ -spaces, a natural, extensive, and completely unstudied class of spaces.

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\* That is, at each point  $x \in X$  there is a neighborhood whose closure is a  $k$ -space.

### § 3. $k'$ -spaces.

**III. Definition 4.** A topological space  $X$  is called a  $k'$ -space if from  $q \in [Q]$ , where  $q$  and  $Q$  are an arbitrary point and set in the space  $X$ , it follows that there exists a bicomact set  $\Phi \subseteq X$  such that  $q \in [Q \cap \Phi]$ . The  $k'$ -spaces include, for example, all Fréchet spaces (see <sup>(4)</sup>).

The theory of  $k$ - and  $k'$ -spaces makes it possible to obtain new results on complete topological spaces. First of all:

**Theorem 12.** *Every complete topological space is a  $k$ -space.*

In fact, a more general assertion holds. Let  $F$  be a bicomactum and let  $X_\alpha$  be arbitrary subsets of it of type  $G_\delta$  in  $F$ ,  $\alpha \in M$ . Then the space

$$X = \bigcup_{\alpha \in M} X_\alpha$$

is a  $k$ -space.

From Theorems 12 and 7 we obtain

**Corollary 1.** *A locally complete space is a  $k$ -space.*

From Theorems 11 and 12 it follows:

**Corollary 2.** *A complete topological space  $X$  is paracompact if and only if into each of its covers one can inscribe a cover that is finite on every bicomactum from  $X$  ("bicomactly finite").*

**Theorem 13.** *Under quasi-open  $k$ -mappings a complete space is carried into a complete one, and a complete paracompact space into a complete paracompact one.*

**Remark 1.** The property of a space of being a  $k'$ -space, as in the case of  $k$ -spaces, is hereditary with respect to closed subsets, and in regular spaces also with respect to open subsets. The following is important.

**Theorem 14.** *A space that is hereditarily a  $k$ -space is a  $k'$ -space.*

**Corollary 1.** *Every space with the first axiom of countability is a  $k'$ -space.*

**Corollary 2.** *Every metric space is a  $k'$ -space.*

By Remark 1, a locally bicomact space is a  $k'$ -space.

A quasi-open image of a  $k'$ -space, generally speaking, need not be a  $k'$ -space. But the following holds.

**Theorem 15.** *A pseudo-open\* image of a space with the first axiom of countability is a  $k'$ -space.*

**Corollary 1.** *A pseudo-open image of a metric space is a  $k'$ -space.*

These results are supplemented by

**Theorem 16.** *A closed image of a  $k'$ -space is a  $k'$ -space.*

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### CITED LITERATURE

<sup>1</sup> P. Alexandrov, *Bull. Acad. Polon. Sci.*, ser. Math., **8**, No. 3 (1960).

<sup>2</sup> V. Ponomarev, *ibid.*

<sup>3</sup> J. L. Kelley, *General Topologie*, N. Y., 1955.

<sup>4</sup> P. S. Urysohn, *Tr. po topologii i drugim oblastyam matematiki*, **2**, 1951.

<sup>5</sup> D. E. Cohen, *Quart. J. Math.*, Oxford ser., (2), **5**, 77 (1954).

\* A mapping  $f : X \rightarrow Y$  is called pseudo-open if for every  $y \in Y$  and every open subset  $U \subseteq f^{-1}y$  of  $X$ , one necessarily has  $\text{Int}(fU) \in y$  (the interior of the set  $fU$  contains  $y$ ).

*Note: Figure translations are in progress. See original paper for figures.*

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