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1963

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Abstract

Full Text

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ON THE SPECTRUM OF TOEPLITZ MATRICES

(Presented by Academician P. S. Aleksandrov, 23 X 1962)

1. Let A be a self-adjoint operator in a Hilbert space H, having simple spectrum; $g \in D_A$ be some generating vector; P be the operator of orthogonal projection onto the subspace $Hg = H \ominus g$. In Hg construct the operator by the formula $Agx = PAx$ for $x \in D_A \cap Hg$. At the beginning of the note the spectral properties of Ag are studied.

If A has discrete spectrum (we do not consider this case), then a complete description of the spectrum of Ag is given by the following proposition of M. G. Krein: if λ_i are the eigenvalues of A, e_i the eigenvectors and $g = \sum a_i e_i$, then the spectrum of Ag coincides with the set of roots of the equation

$$\sum \frac{|a_i|^2}{\lambda_i - z} = 0.$$

We begin with the following remark. Let $\sigma(t)$ be a nondecreasing function on $(-\infty, \infty)$ and $\int t^2\, d\sigma(t) < \infty$. Put

$$f(z) = \int \frac{d\sigma(t)}{t - z} \qquad (\operatorname{Im} z > 0). \tag{1}$$

Obviously, $\operatorname{Im}\left(-\frac{1}{f(z)}\right) \geq 0$ for $\operatorname{Im} z > 0$; using the well-known theorem of Herglotz (([1]), p. 117) and the condition $\int t^2\, d\sigma(t) < \infty$, it is easy to show that

$$-\frac{1}{f(z)} = \gamma z + \beta + \int \frac{d\sigma^*(t)}{t - z}, \tag{2}$$

where $\sigma^*(t)$ is a nondecreasing function, uniquely determined by the function $\sigma(t)$ by formulas (1) and (2). We now pass to the description of the spectrum of Ag.

Theorem 1. *Let E_t be the spectral family of the operator A, $\sigma(t) = (E_{tg}, g)$, and let $\sigma^*(t)$ be determined from $\sigma(t)$ by formulas (1) and (2). Then Ag is unitarily equivalent to the operator of multiplication by the independent variable in the space $L_2(\sigma^*, -\infty, \infty)$.*

As is known, every spectral family E_t in H is uniquely decomposed into a sum $E_t^a + E_t^c$, where E_t^a is a weakly absolutely continuous function, and E_t^c is a weakly singular* function of t_0.

Theorem 2. *Let E_t and F_t be the spectral families of the operators A and Ag; E_t^a and F_t^a their absolutely continuous, and E_t^c and F_t^c their singular components. Then E_t^a and F_t^a are unitarily equivalent, and E_t^c and F_t^c are mutually singular.*

Let us explain that the mutual singularity of E_t^c and F_t^c means the mutual singularity of the scalar measures corresponding to the functions $(E_t^c h, h)$ and $(F_t^c g, g)$ for any $h \in H$, $g \in Hg$.

We outline the proofs of the theorems.

* This means that for any $h \in H$ the measure $d(E_t^a h, h)$ is absolutely continuous, while $d(E_t^c h, h)$ is singular; here the singular function is not necessarily discontinuous.

One may assume that $H = L_2(\sigma)$, $g = g(t) \equiv 1$, $Af = tf(t)$. Let first $\|A\| < \infty$, and hence $d\sigma$ is finite. In the basis consisting of the orthogonal polynomials $P_n(t)$ $(n \geqslant 0)$ with respect to $d\sigma$, the operator A is represented by the Jacobi matrix J, and the operator Ag by the matrix J_1, obtained from J by deleting the first row and the first column. From the matrix J_1 we construct in the known way the measure $d\sigma^*$ and the orthogonal polynomials $P_n^*(t)$ $(n \geqslant 0)$. Then the correspondence $P_n(t) \leftrightarrow P_{n+1}^*(t)$ gives rise to an isometric correspondence U between $L_2(\sigma) \ominus g$ and $L_2(\sigma^*)$, under which Ag goes over into multiplication by t in $L_2(\sigma^*)$. Constructing the functions $f(z) = \int \frac{d\sigma(t)}{t-z}$ and $f^*(z) = \int \frac{d\sigma^*(t)}{t-z}$, and, by the known formula for the expansion of these functions into continued fractions ((1), p. 34), finding

$$f^*(z) = \gamma z + \beta - \frac{1}{f(z)},$$

we obtain formulas (1) and (2), and this proves the theorem for the case $\|A\| < \infty$.

From the properties of orthogonal polynomials it is easy to obtain that the mapping U introduced above is given by the formula

$$Uf = \int \frac{f(x) - f(t)}{x - t}\, d\sigma(t) \tag{3}$$

and that

$$U^{-1}\varphi = \int \frac{\varphi(x) - \varphi(t)}{x - t}\, d\sigma^*(t) + (x - a_0)\varphi(t). \tag{4}$$

To prove the theorem in the case $\|A\| = \infty$, we first prove, by passage to the limit from finite measures, that formulas (3) and (4) establish an isometric correspondence between $L_2(\sigma) \ominus g$ and $L_2(\sigma^*)$; then we prove that U takes Ag into multiplication by t in $L_2(\sigma^*)$.

To prove Theorem 2, put $\operatorname{Im} z \to 0$ in (1) and (2). Then

$$\frac{d\sigma(x)}{dx} = |f(x+i0)|^2 \frac{d\sigma^*(x)}{dx}.$$

Therefore $0 < \sigma'(x)/\sigma^{*'}(x) < \infty$ almost everywhere, which is equivalent to the first assertion of Theorem 2.*

From the easily proved inequalities**

$$\operatorname{Im} f(z) > \frac{\sigma(x+y)-\sigma(x-y)}{2y}, \qquad \operatorname{Im} f^*(z) > \frac{\sigma^*(x+y)-\sigma^*(x-y)}{2y}$$

and (1), (2), we obtain

$$\frac{\sigma(x+y)-\sigma(x-y)}{2y}\,\frac{\sigma^*(x+y)-\sigma^*(x-y)}{2y} < 1.$$

From this inequality one easily obtains the second assertion of Theorem 2.

II. Let us apply Theorem 2 to Toeplitz matrices. If $F(x) \neq \text{const}$, $F(x) \in L_2(-\pi,\pi)$ and $F(x) = \sum c_n e^{inx}$, then the matrix $\{a_{ij}\} = \{c_{i-j}\}$ $(i, j \geqslant 0)$ is called a Toeplitz matrix. In the space l_2 of one-sided sequences $x = \{x_k\}$ $(k \geqslant 0)$, define the operator T_F^0, setting, for finite

$$x = \{x_k\} \quad T_F^0 x = y, \quad \text{where } y = \{y_n\}, \quad y_n = \sum_0^\infty c_{n-j} x_j \quad (n \geqslant 0).$$

Let T_F be the closure of the operator T_F^0. The spectral properties of the operator T_F were studied in (3,5,8).

The principal known results are as follows:

1) If $F(x)$ is semibounded, then T_F is self-adjoint (2).
2) If T_F is self-adjoint, then its spectrum is continuous and fills the interval $(\inf F, \sup F)$ (2).
3) If $F(x)$ is semibounded, then T_F has an absolutely continuous spectrum (8).
4) If $F(x)$ is an even periodic function, $F'(x)$ exists and is expandable in an absolutely convergent Fourier series, then T_F is unitarily equivalent to the absolutely continuous component of the operator of multiplication by $F(x)$ in $L_2(0,\pi)$ (8).

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* This assertion also follows from Kato' s theorem on finite-dimensional perturbations (4).

** A similar inequality is found in Fatou (3).

Below we give a complete spectral description of the operator T_F for arbitrary $F(x) \in L_2(-\pi, \pi)$.

Theorem 3. *The operator T_F has an absolutely continuous spectrum if and only if it is self-adjoint.*

To clarify the idea of the proof, let us consider the simplest case: suppose T_F has simple spectrum and the vector $e_0 = (1, 0, 0, \ldots)$ is a generating vector. Let $\widehat{T}_F$ be defined in $H = l_2 \ominus e_0$ by the formula $\widehat{T}_F x = PT_{Fx}$, where P is the projection onto H. Obviously, T_F and $\widehat{T}_F$ are unitarily equivalent; but, by Theorem 2, their singular components are mutually singular. Thus T_F cannot have a singular component. In the general case the proof also uses Theorem 2, but requires rather cumbersome geometric considerations.

For what follows we shall need the following.

Definition 1. Let $N_F(\lambda)$ be a function of λ defined as follows: if the set $E\{x : F(x) < \lambda\}$ consists of a finite number of intervals mod 0, then $N_F(\lambda)$ is equal to the number of these intervals; otherwise $N_F(\lambda) = \infty$.

In defining $N_F(\lambda)$ one should take into account that $F(x)$ is considered on the interval $(-\pi, \pi)$ with the endpoints $-\pi$ and π identified. Note that for smooth functions $F(x)$, the function $2N_F(\lambda)$ coincides with the Banach indicatrix of the function $F(x)$. The following theorem gives a complete spectral description of the operator T_F.

Theorem 4. *Let E_m be the set on the λ-axis where $N_F(\lambda) = m$ $(m = 1, 2, \ldots, \infty)$, and let A_m be the operator of multiplication by an independent variable in $L_2(E_m)$. Further, let $B_m = A_m \oplus \cdots \oplus A_m$ (the sum contains m copies of the operator A_m) and $A_F = B_1 \oplus \cdots \oplus B_\infty$. Then T_F is unitarily equivalent to the operator A_F.*

Let us consider the simplest case: $N_F(\lambda) = 1$ for $\inf F < \lambda < \sup F$.

Map $l_2(0, \infty)$ onto the space H_2 of functions $f(z)$,

$$f(z) = \sum_0^\infty a_n z^n$$

$(|z| < 1)$, with metric

$$\|f\|^2 = \sum_0^\infty |a_n|^2.$$

Let $e_k = z^k$ and $R_\lambda = (T_F - \lambda E)^{-1}$. One can find

$$\chi_k(\sigma, z) = \lim_{\varepsilon \to 0} (R_{\sigma + i\varepsilon} e_k - R_{\sigma - i\varepsilon} e_k).$$

It turns out that $\chi_k(\sigma, z) = p_k(\sigma)\chi_0(\sigma, z)$, and $\chi_k(\sigma, z) \neq 0$ for almost all σ. Hence it follows that T_F has a simple Lebesgue spectrum. An analogous consideration is used for the proof in the general case.

For some functions $F(x)$, the description of the spectrum of the operator T_F can be carried out without Theorem 4; for this purpose the following theorem is used:

Theorem 5. *If A, B, and $C = A + B$ are self-adjoint bounded operators in H; A^a, B^a, C^a are their absolutely continuous components, and the operator AB is nuclear, then C^a is unitarily equivalent to $A^a \oplus B^a$.*

It is easy to show that if $\Phi_1(x)$ and $\Phi_2(x)$ are functions concentrated on disjoint closed sets, then $T_{\Phi_1} \cdot T_{\Phi_2}$ is a nuclear operator and, by Theorem 5, $T_{\Phi_1+\Phi_2}$ is equivalent to $T_{\Phi_1} \oplus T_{\Phi_2}$. Repeated application of this remark to the operator T_F makes it possible in a number of cases to reduce the problem of the spectrum to the simplest case, when the operator has a simple spectrum (see above).

Theorem 6. *The operator T_F is self-adjoint if, in some neighborhood of each point $x \in (-\pi, \pi)$ of the interval (with the endpoints $-\pi$ and π identified), the function $F(x)$ is semibounded.*

Finally, let us note that for two extensions A_1 and A_2 of a simple symmetric operator A, a theorem analogous to Theorem 2 can be proved: the singular components A_1 and A_2 are mutually singular, while the absolutely continuous ones are equivalent. From this one can obtain Putnam’ s results (6,7).

Note added in proof. After the present paper had been submitted for publication, I learned that the result concerning the spectra of two extensions of a symmetric operator had been obtained earlier by Aronszajn (9), although only as applied to the Sturm-Liouville operator.

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Received
19 X 1962

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