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Reports of the Academy of Sciences of the USSR

V. N. GORCHAKOV

1963

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Abstract

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Reports of the Academy of Sciences of the USSR

1963. Volume 152, No. 3

MATHEMATICS

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ON THE ASYMPTOTICS OF THE SPECTRAL FUNCTION OF A CLASS OF HYPOELLIPTIC OPERATORS

(Presented by Academician S. L. Sobolev on 10 IV 1963)

1°. In the present note, using Carleman's method ⁽¹⁾, we study the asymptotics of the spectral function of a certain class of hypoelliptic operators ⁽²⁾, p. 88, inside a domain.

For general elliptic operators such asymptotics were obtained by Gårding ⁽³⁾ and Browder ⁽⁴⁾, and an estimate of the remainder term by Gårding ⁽⁵⁾; for the development of these methods see Bergendal ⁽⁶⁾. More refined results for the Laplace operator with the Dirichlet boundary condition were obtained by another method by B. M. Levitan ⁽⁷⁾. As far as we know, the study of the asymptotics of the spectral function of hypoelliptic operators has not previously been carried out.

2°. Let D denote a bounded domain of the Euclidean plane R_2 ; let $x = (x_1, \dots, x_n)$ be a point of the n -dimensional Euclidean space R_n with scalar product $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$; let $C_0^\infty(D)$ be the set of finite infinitely differentiable functions in D ; and let $C_0^k(D)$ be the set of finite functions in D having k continuous derivatives.

On functions $u \in C_0^\infty(D)$ consider a hypoelliptic differential operator in two variables of the form

$$P\left(\frac{1}{i} \frac{\partial}{\partial x}\right) = (-1)^r a \frac{\partial^{2r} u}{\partial x_1^{2p_1} \partial x_2^{2p_2}} + (-1)^{m_1} b \frac{\partial^{2m_1} u}{\partial x_1^{2m_1}} + (-1)^{m_2} c \frac{\partial^{2m_2} u}{\partial x_2^{2m_2}} \quad (1)$$

with constant real coefficients $a, b, c > 0$; $P(s) = P(s_1, s_2)$ is the characteristic polynomial of the operator, $p_1 < m_1$, $p_2 < m_2^*$. The closure of the operator P in the space $L_2(D)$ —the so-called minimal operator ⁽²⁾—is a symmetric nonnegative

operator. The spectral family of any self-adjoint nonnegative extension \widehat{P} of it is an integral operator

$$E_\lambda f(y) = \int_D \theta(x, y, \lambda) f(x) dx$$

with Carleman kernel (see (2), theorem (3,9)). We shall be interested in the asymptotics of the kernel $\theta(x, y, \lambda)$ —the spectral function of the operator \widehat{P} —as $\lambda \rightarrow +\infty$. If $D = R_n$, then the minimal operator P_0 is self-adjoint. Its spectral function depends on the difference and is expressed by the formula

$$\theta_0(x - y, \lambda) = \frac{1}{(2\pi)^n} \int_{P(s) \leq \lambda} e^{-i(x-y, s)} ds. \quad (2)$$

* The last condition follows from the definition of hypoellipticity of the operator.

We now formulate the main results in the form of the following theorems:

Theorem 1. *For hypoelliptic operators of the form (1), the asymptotics of the function $\theta_0(x - y, \lambda)$ for $x = y^*$ ($\lambda \rightarrow +\infty$) is expressed by the formulas*

$$\theta_0(0, \lambda) \sim \begin{cases} A\lambda^{\frac{m_1+p_2-p_1}{2m_1p_2}}, & p_1 < p_2, \\ B\lambda^{\frac{1}{2p}} \ln \lambda, & p_1 = p_2 = p, \\ C\lambda^{\frac{m_2+p_1-p_2}{2m_2p_1}}, & p_1 > p_2 \end{cases} \quad (3)$$

when

$$\frac{p_1}{m_1} + \frac{p_2}{m_2} > 1;$$

$$\theta_0(0, \lambda) \sim K\lambda^{\frac{1}{2m_1} + \frac{1}{2m_2}} \quad (4)$$

when

$$\frac{p_1}{m_1} + \frac{p_2}{m_2} \leq 1.$$

A, B, C, K are certain constants depending on m_1, m_2, p_1, p_2 .

Theorem 2. *Let $\theta(x, y, \lambda)$ be the spectral function of any self-adjoint nonnegative extension \widehat{P} of a hypoelliptic operator of the form (1). Then the following asymptotic estimates hold as $\lambda \rightarrow +\infty$:*

$$\theta(x, y, \lambda) = \theta_0(x - y, \lambda) + \begin{cases} o\left(\lambda^{\frac{m_1+p_2-p_1}{2m_1p_2}}\right), & p_1 < p_2, \\ o\left(\lambda^{\frac{1}{2p}} \ln \lambda\right), & p_1 = p_2 = p, \\ o\left(\lambda^{\frac{m_2+p_1-p_2}{2m_2p_1}}\right), & p_1 > p_2 \end{cases} \quad (5)$$

when

$$\frac{p_1}{m_1} + \frac{p_2}{m_2} > 1;$$

$$\theta(x, y, \lambda) = \theta_0(x - y, \lambda) + o\left(\lambda^{\frac{1}{2m_1} + \frac{1}{2m_2}}\right) \quad (6)$$

when

$$\frac{p_1}{m_1} + \frac{p_2}{m_2} \leq 1.$$

The estimate o is uniform with respect to $(x, y) \in W \times W$, where W is any compact subset of the domain D .

3°. We outline the proof of the first assertion. Instead of the function $\theta_0(x, \lambda)$ itself, consider its Laplace-Stieltjes transform with respect to λ (which, as is known, is at the same time the fundamental solution of the Cauchy problem for the generalized parabolic** equation $\frac{\partial u}{\partial t} = P\left(\frac{1}{i} \frac{\partial}{\partial x}\right) u$, i.e.

$$G_0(t, x) = \int_0^\infty e^{-t\lambda} d\theta_0(x, \lambda) = \frac{1}{(2\pi)^2} \int_{R_2} e^{-tP(s) - i\langle x, s \rangle} ds \quad (7)$$

and find its asymptotics as $t \downarrow 0$, $x = 0$.

Thus, the matter reduces to studying the behavior of the double integral (7) as $t \downarrow 0$, $x = 0$.

* From formula (2) it is clear that $\theta_0(0, \lambda)$ is proportional to the area bounded by the level lines of the polynomial $P(s)$.

** In the sense of G. E. Shilov (8).

***) The multiple integral over R_2 converges absolutely by virtue of the estimate $P(s) > c|s|^m$, $m = \min(m_1, m_2)$.

It can be shown that for operators of the form (1)

$$G_0(t, 0) \sim \begin{cases} A_1 t^{-\frac{m_1+p_2-p_1}{2m_1p_2}}, & p_1 < p_2, \\ B_1 t^{-\frac{1}{2p}} \ln t^{-1}, & p_1 = p_2 = p, \\ C_1 t^{-\frac{m_2+p_1-p_2}{2m_2p_1}}, & p_1 > p_2, \end{cases} \quad (8)$$

when

$$\frac{p_1}{m_1} + \frac{p_2}{m_2} > 1;$$

$$G_0(t, 0) \sim K_1 t^{-\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)}, \quad (9)$$

when

$$\frac{p_1}{m_1} + \frac{p_2}{m_2} \leq 1.$$

Taking into account the last relations and the fact that $\theta_0(0, \lambda)$ is a nondecreasing function, and applying Karamata's Tauberian theorem for the Laplace transform ((9), Theorem I–II), we obtain the required assertion.

4°. To prove Theorem 2, consider in $L_2(D)$ the Hermitian bilinear form

$$V(t, f, g) = ((\exp(-t\hat{P}) - \exp(-tP_0))f, g).$$

Its kernel is the difference

$$v(t, x, y) = G(t, x, y) - G_0(t, x - y),$$

where $G(t, x, y)$ denotes the Laplace-Stieltjes transform of the spectral function $\theta(x, y, \lambda)$ of the extension \hat{P} (the existence of $G(t, x, y)$ also follows from Hörmander's results ((²), Theorem 3.9)).

Since $V(t, f, g) \rightarrow 0$ as $t \downarrow 0$, it is not difficult to show that, for $t < 1$,

$$\begin{aligned} |V(t, f, g)| &= \left| \int_{D \times D} v(t, x, y) f(x) g(y) dx dy \right| \leq \\ &\leq C_F \sup_{x \in F} |f(x)| \sup_{y \in F} |g(y)| \end{aligned} \quad (10)$$

for any $f, g \in C_0(F)$, where F is an arbitrary compact subset of the domain D , and the constant C_F does not depend on t . We apply Gårding's integral identity (⁵), which, as is easy to see, is also valid in the case of hypoelliptic operators:

$$\begin{aligned}
 G(t, x, y) - G_0(t, x - y) &= \\
 &= \int_0^t dt' \int_0^{t'} dt'' \int_{D \times D} [G(t'', \xi, \eta) - G_0(t'', \xi - \eta)] \times \\
 &\quad \times \left[P \left(\frac{1}{i} \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial t} \right] [G_0(t - t', x - \xi)(1 - h(\xi))] \times \\
 &\quad \times \left[P \left(\frac{1}{i} \frac{\partial}{\partial \eta} \right) + \frac{\partial}{\partial t'} \right] [G_0(t' - t'', y - \eta)(1 - h(\eta))] d\xi d\eta, \quad (11)
 \end{aligned}$$

where $(x, y) \in U \times U$; $h(x) \in C_0^\infty(D)$; F is the support of h ; $h(x) = 1$ for $x \in U$; U is any compact subset of D containing W .

In view of inequality (10) and the last identity, the estimate of the difference

$$G(t, x, y) - G_0(t, x - y)$$

reduces to estimating the derivatives with respect to x and t of the function $G_0(t, x)$ for $|x| > \varepsilon$. This estimate can be obtained by applying repeated integration by parts to the integral (7), and the estimate improves with each integration.*

* We note that for the fundamental solution $G_0(t, x)$ even of such a simple hypoelliptic operator as $Pu = \frac{\partial}{\partial x_1^2}$

In conjunction with identity (11), this gives the following estimate, valid for any integer $k > 0$:

$$|G(t, x, y) - G_0(t, x - y)| \ll \begin{cases} C_k t^{\alpha k + \beta}, & \frac{p_1}{m_1} + \frac{p_2}{m_2} > 1, \\ B_k t^{\frac{k}{\nu} + \gamma}, & \frac{p_1}{m_1} + \frac{p_2}{m_2} \ll 1, \end{cases} \quad (12)$$

where

$$\alpha = \min \left[\frac{1}{2p_1} \left(1 - \frac{p_2}{m_2} \right), \frac{1}{2p_2} \left(1 - \frac{p_1}{m_1} \right) \right] > 0, \quad \nu = \max(m_1, m_2);$$

β and γ are certain negative rational numbers. The constants C_k and B_k depend on the set F . The estimate is uniform with respect to $(x, y) \in W \times W$. It is clear that, for sufficiently large k , the exponents $\alpha k + \beta$ and $k/\nu + \gamma$ are positive.

Now we apply the Tauberian theorem of Ganelius with remainder estimate (10) (see also Bergendal (6)) to the function $\sigma(\lambda) = \theta(x, y, \lambda) - \theta_0(x - y, \lambda)$. To obtain the Tauberian condition, one must first apply the same theorem to the function $\theta_0(0, \lambda) - \theta(x, x, \lambda)$, using formulas (3), (4) of Theorem 1. Then, with the aid of the inequality

$$\text{Var } \theta(x, y, \lambda) \ll \{\text{Var } \theta(x, x, \lambda) \text{Var } \theta(y, y, \lambda)\}^{1/2},$$

one can obtain the condition itself. As a result we arrive at the asymptotic estimates (5), (6).

5°. Let us note that the asymptotics of the spectral function of a hypoelliptic operator of the form

$$Pu = (-1)^{m_1} a \frac{\partial^{2m_1} u}{\partial x_1^{2m_1}} + (-1)^{m_2} b \frac{\partial^{2m_2} u}{\partial x_2^{2m_2}}$$

is obtained trivially, by separation of variables.

It can be shown that for operators of the form (1) an exponential estimate is in fact valid. In the presence of such an estimate, the Ganelius theorem gives a more precise result, namely: the Riesz means of sufficiently high order of the function $\theta(x, y, \lambda)$, as $\lambda \rightarrow +\infty$, will tend to the corresponding Riesz means of the function $\theta_0(x - y, \lambda)$.

In conclusion, the author expresses deep gratitude to B. I. Korenblyum and Yu. M. Berezanskii for their help and valuable comments.

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Received
6 IV 1963

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Note: Figure translations are in progress. See original paper for figures.

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