# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

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# ON THE CAUCHY PROBLEM FOR PARABOLIC EQUATIONS WITH A SMALL PARAMETER

*(Presented by Academician S. L. Sobolev, VII 5, 1962)*

Consider the solution $u^\varepsilon(x,t)$ of the $2p$-parabolic [1] equation

$$\widetilde{L}_\varepsilon u^\varepsilon(x,t) \equiv L_\varepsilon^1 u^\varepsilon(x,t) + L_\varepsilon^2 u^\varepsilon(x,t) + L_0 u^\varepsilon(x,t) = 0, \tag{1}$$

satisfying the condition

$$u^\varepsilon(x,t)\big|_{t=0} = \Psi(x) \equiv \psi(x)\chi(x_1), \tag{2}$$

where

$$L_\varepsilon^1 u^\varepsilon = (-1)^p \sum_{s\le|i|\le 2p} a_i(x,t)\varepsilon^{|i|/2p} D^i u^\varepsilon;$$

$$L_\varepsilon^2 u^\varepsilon \equiv \varepsilon^{s/2p} \sum_{1\le|i|\le s-1} a_i(x,t) D^{|i|} u^\varepsilon;$$

$$L_0 u^\varepsilon(x,t) = \sum b_i(x,t)\frac{\partial u^\varepsilon}{\partial x_i} + \frac{\partial u^\varepsilon}{\partial t} + c(x,t)u^\varepsilon(x,t);$$

$$\varepsilon > 0; \quad s \text{ is an integer, } 1 \le s \le 2p-1; \quad i = (i_1, \dots, i_n), \quad |i| = i_1 + \dots + i_n;$$

$$x = (x_1, \dots, x_n) \in E^n \ (-\infty < x_i < \infty); \quad D^i = \partial^{|i|}/\partial x_1^{i_1} \cdots \partial x_n^{i_n}, \quad t \in [0,T];$$

$\chi(x_1)$ is equal to zero for $x_1 < 0$ and to one for $x_1 \ge 0$. For simplicity of exposition we shall assume that all coefficients in (1) and the function $\psi(x)$ belong to $C^\infty(E^n \times [0,T])$ and are bounded in $E^n \times [0,T]$ together with all their derivatives.

We shall be interested in the behavior of the solution $u^\varepsilon(x,t)$ of problem (1)–(2) as $\varepsilon \to 0$. In the case $p = 1$ this problem was considered in [2,3]. Denote by $u^0(x,t)$ the solution of problem (1)–(2) for $\varepsilon = 0$ (we shall call it problem $(1^0)$—$(2^0)$), and by $l(x,t)$ the characteristic of equation $(1^0)$ passing through the point $(x,t)$. The solution $u^0(x,t)$ has a discontinuity of the first kind along the characteristics $l(x,t)$ passing through the plane $x_1 = t = 0$. These characteristics will be called discontinuity characteristics, and the surface which they form will be called the discontinuity surface.

**Theorem 1.** *As $\varepsilon \to 0$, the solution $u^\varepsilon(x,t)$ of problem (1)–(2) converges to the solution $u^0(x,t)$ of problem $(1^0)$—$(2^0)$ everywhere outside the discontinuity surface of the function $u^0(x,t)$. This convergence is uniform in $(x,t) \in E^n \times [0,T]$ outside any $\delta$-neighborhood of the discontinuity surface of the function $u^0(x,t)$, $\delta > 0$.*

In what follows we shall assume that the operator $L_0$ has the form $L_0 u = \partial u/\partial t + c(x,t)u$ (i.e., in (1) $b_i(x,t) = 0$). This can be achieved by replacing $(x,t)$ by $(\bar{x},\bar{t})$, where $\bar{t}$ is the length of the arc of the characteristic $l(x,t)$ between the points $(\bar{x},0)$ and $(x,t)$, and $\bar{x} = (\bar{x}_1,\dots,\bar{x}_n)$ are the coordinates of the point of intersection of $l(x,t)$ with the plane $t = 0$.

I. Let $n = 1$. Then problem (1)–(2) takes the form:

$$\widetilde{L}_\varepsilon(x,t)u^\varepsilon(x,t) \equiv \varepsilon a_{2p}(x,t)\frac{\partial^{2p}u^\varepsilon}{\partial x^{2p}} + \dots + \varepsilon^{s/2p}a_s(x,t)\frac{\partial^s u^\varepsilon}{\partial x^s} +$$

$$+\varepsilon^{s/2p}\left\{a_{s-1}(x,t)\frac{\partial^{s-1}u^\varepsilon}{\partial x^{s-1}} + \dots + a_1(x,t)\frac{\partial u^\varepsilon}{\partial x}\right\} + \frac{\partial u^\varepsilon}{\partial t} + c(x,t)u^\varepsilon = 0, \quad (3)$$

$$u^\varepsilon(x,t)\big|_{t=0} = \psi(x)\chi(x), \quad (4)$$

where $x \in E'(-\infty < x < \infty)$, $t \in [0,T]$, $a_{2p}(x,t) = (-1)^p\tilde{a}_{2p}(x,t)$, $\tilde{a}_{2p}(x,t) \geqslant \beta \geqslant 0$.

Put $\widetilde{L}_\varepsilon(x,t) \equiv L_{1\varepsilon}(x,t)+L_0(x,t)$, where $L_0(x,t) \equiv \partial/\partial t + c(x,t)$, $L_{1\varepsilon} = L^1_\varepsilon + L^2_\varepsilon$, $\alpha(x,t) = \exp\left\{-\int_0^t c(x,\tau)\,d\tau\right\}$, and denote by $U_0^\varepsilon(x,\xi,t,\tau)$ the fundamental solution of equation (3) in the case when the coefficients do not depend on $x$, $c(x,t) = 0$, $L^2_\varepsilon = 0$. Consider the functions

$$v^\varepsilon_{kj}(x,t,\Phi_\varepsilon(0,t)) \equiv v^\varepsilon_{kj}(\Phi_\varepsilon) = \varepsilon^{j/2p}\int_0^t a_j\frac{\partial^k\Phi_\varepsilon(0,t)}{\partial x^k}\frac{\partial^{j-1-k}U_0^\varepsilon(x,0,t,\tau)}{\partial x^{j-1-k}}\,d\tau$$

$$(k = 0,1,\dots,j-1;\ j = 1,\dots,2p),$$

where $\Phi_\varepsilon(x,t)$ is a function infinitely differentiable with respect to $x, t$ and uniformly bounded in $\varepsilon$, together with all its derivatives with respect to $x, t$, for $(x,t) \in E' \times [0,T]$.

**Lemma.** If for all $(x,t) \in E' \times [0,T]$

$$|\Phi_\varepsilon(x,t)| \leqslant C\varepsilon^{l/2p},$$

where the constant $C > 0$, then

$$|v^\varepsilon_{kj}(\Phi_\varepsilon)| \leqslant C_1 t\,(\varepsilon t)^{k/2p}\varepsilon^{l/2p} \quad \left(\text{i.e. } v^\varepsilon_{kj} = O\left(\varepsilon^{(l+k)/2p}\right)\right) \tag{5}$$

$$(k = 0, 1, \dots, j-1;\ j = 1, \dots, 2p)$$

for $(x,t) \in E' \times [0,T]$, with a constant $C_1 > 0$.

The proof of this lemma is based on estimates of the fundamental solution $U^\varepsilon_0(x, \xi, t, \tau)$ obtained in (4).

The functions $v^\varepsilon_{kj}(\Phi_\varepsilon)$ defined above ($k = 0, 1, \dots, j-1;\ j = 1, \dots, 2p$), for which the relations (5) are valid, will be called a $2p$-parabolic boundary layer of order $(k+l)$.

**Theorem 2.** If the coefficients of equation (3) do not depend on $x$, $L^2_\varepsilon = 0$, then the solution $u^\varepsilon(x,t)$ of problem (3)–(4) can be represented in the form

$$u^\varepsilon(x,t) = u^0(x,t) +$$

$$+\alpha(t)\left\{\sum_{i=0}^{l}\sum_{j=s}^{2p}\sum_{k=0}^{j-1} v^\varepsilon_{kj}(z^{i\varepsilon}) + \sum_{i=1}^{l}\chi(x) z^{i\varepsilon}(x,t) + \bar{z}^{(l+1)\varepsilon}(x,t)\right\}, \tag{6}$$

where $u^0(x,t)$ is the solution of problem (3)–(4) for $\varepsilon = 0$; $z^{0\varepsilon}(x,t) = \psi(x)$; $z^{i\varepsilon}(x,t)$, $i = 1, 2, \dots, l$, are determined recurrently by

$$\frac{dz^{(i+1)\varepsilon}}{dt} = -L_{1\varepsilon}(t) z^{i\varepsilon}(x,t), \qquad z^{(i+1)\varepsilon}\big|_{t=0} = 0, \qquad i = 0, 1, \dots, l-1;$$

$$[L_{1\varepsilon}(t) + \partial/\partial t]\bar{z}^{(l+1)\varepsilon}(x,t) = -[L_{1\varepsilon}(t) z^{l\varepsilon}(x,t)]\chi(x), \qquad \bar{z}^{(l+1)\varepsilon}\big|_{t=0} = 0.$$

Moreover,

$$z^{i\varepsilon}(x,t) = O(\varepsilon^{is/2p}), \qquad i = 1, \dots, l, \qquad \bar{z}^{(l+1)\varepsilon}(x,t) = O(\varepsilon^{(l+1)s/2p})$$

for $(x,t) \in E'[0,T]$; $v_{kj}^{\varepsilon}(z^{i\varepsilon})$ $(k=0,1,\dots,j-1;\ j=1,\dots,2p)$ are functions of a $2p$-parabolic boundary layer of order $(k+is)$.

If $L_\varepsilon^2 \neq 0$, then instead of (6) the following representation will hold for $u^\varepsilon(x,t)$:

$$u^\varepsilon(x,t) = w_0^\varepsilon + w_1^\varepsilon + \dots + w_m^\varepsilon + O\left(\varepsilon^{\frac{m+1}{2p}}\right), \tag{6'}$$

where (6) is valid for $w_0^\varepsilon$, while $w_{k+1}^\varepsilon = O(\varepsilon^{k/2p})$, $k=0,\dots,m-1$, are determined recursively:

$$(L_\varepsilon^1 + L_0)w_{k+1}^\varepsilon = -L_\varepsilon^2 w_k^\varepsilon, \qquad w_{k+1}^\varepsilon\big|_{t=0} = 0.$$

We note that from (6) the validity of Theorem 1 in the case under consideration follows immediately.

**Theorem 3.** If the coefficients of equation (3) depend on $x$ and $t$, then for the solution $u^\varepsilon(x,t)$ of problem (3)–(4) the representation

$$u^\varepsilon(x,t) = \alpha(x,t)\{\overline{u}^0(x,t) + \overline{w}_0^\varepsilon(x,t) + w_1^\varepsilon + \dots + w_m^\varepsilon\} + O(\varepsilon^{(m+1)/2p}),$$

holds, where $\alpha(x,t)\overline{u}^0(x,t) = u^0(x,t)$. Moreover, for $u^0 + \overline{w}_0^\varepsilon$ in an $\varepsilon^{1/2p}$-neighborhood of $x=0$ the representation $(6')$ is valid, while $w_k^\varepsilon(x,t) = O(\varepsilon^{k/2p})$, $k=1,\dots,m$, uniformly with respect to $(x,t) \in E' \times [0,T]$.

II. Let $n>1$. Denote by $\widetilde{L}_{1\varepsilon}(x,t)$ that part of the operator $\widetilde{L}_\varepsilon(x,t)$ which contains differentiation only with respect to $x_1$, and by $u_1^\varepsilon(x,t)$ the solution of the equation $\widetilde{L}_{1\varepsilon}u_1^\varepsilon(x,t) = 0$ under condition (2) (here $x_2,\dots,x_n$ are regarded as parameters).

**Theorem 4.** For the solution $u^\varepsilon(x,t)$ of problem (1)–(2) there is the representation

$$u^\varepsilon(x,t) = u_1^\varepsilon(x,t) + u_2^\varepsilon(x,t) + \dots + u_{m+1}^\varepsilon(x,t) + O(\varepsilon^{(m+1)/2p}), \tag{7}$$

where $u_k^\varepsilon$, $k=2,\dots,m+1$, are determined recursively:

$$\widetilde{L}_{1\varepsilon}u_{k+1}^\varepsilon(x,t) = -\widetilde{L}_\varepsilon u_k^\varepsilon(x,t), \qquad u_{k+1}^\varepsilon(x,t)\big|_{t=0} = 0, \qquad k=1,\dots,m;$$

moreover, Theorem 3 is valid for $u_1^\varepsilon(x,t)$, and $u_k^\varepsilon(x,t) = O(\varepsilon^{(k-1)/2p})$, $k = 2,\dots,m+1$.

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## CITED LITERATURE


1. I. G. Petrovskii, *Bull. Moscow State Univ.*, Section A, **7** (1938).
2. E. K. Isakova, *Dokl. Akad. Nauk SSSR*, **117**, No. 6 (1957).
3. E. K. Isakova, *Dokl. Akad. Nauk SSSR*, **119**, No. 6 (1958).
4. O. A. Ladyzhenskaya, *Mat. sbornik*, **27** (69), 175 (1950).


*Note: Figure translations are in progress. See original paper for figures.*