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Abstract

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QUALITATIVE DESCRIPTION OF STRONG RESONANCE IN A NONLINEAR SYSTEM

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Let a system be given

$$\dot{x} = \omega y + f(x, y, t), \qquad \dot{y} = -\omega x + g(x, y, t), \tag{1}$$

where $\omega > 0$, and the functions $f(x, y, t)$ and $g(x, y, t)$ are analytic in x, y, and t in some domain $|x| < R_0$, $|y| < R_0$, $|\operatorname{Im} t| < \delta_0$ ($R_0 > 0$, $\delta_0 > 0$) and are periodic in t with period 2π. Suppose further that the expansion of these functions in a series in powers of x and y contains no terms below the second degree. Then the behavior of the trajectories of system (1) in a neighborhood of the point $x = y = 0$ is determined, on the one hand, by the arithmetic nature of the frequency ω, and on the other hand, by certain invariants of system (1), depending on the particular form of the functions $f(x, y, t)$ and $g(x, y, t)$. The effects arising in such a system have been called resonance effects and, until recently, were studied exclusively by means of asymptotic methods or methods close to them. However, the investigation of various cases of the motion of charged particles in a magnetic field showed ([1,2]) that, in order to answer many questions connected with the character of this motion, other methods are needed. The purpose of the present work is to develop another approach to this problem. The method set forth below has made it possible, in particular, to investigate the stability of radial oscillations in the isochronous cyclotron ([2]).

1. Transformation of system (1) to canonical form

In system (1) pass from the variables x, y to the variables $u = x + iy$, $\bar{u} = x - iy$. In the new variables system (1) has the form

$$\dot{u} = -i\omega u + p(u, \bar{u}, t), \qquad \dot{\bar{u}} = i\omega \bar{u} + q(u, \bar{u}, t), \tag{2}$$

where $p(u, \bar{u}, t) = \overline{q(u, \bar{u}, t)}$ can always, by virtue of the assumptions made, be expanded in a convergent series in powers of u and $\bar{u}$. Without loss of generality, we shall assume that $p(u, \bar{u}, t)$ has the form

Fig. 1

Figure 1: Fig. 1

$$p(u,\bar{u},t)=\sum_{\alpha=1}^{k}C_\alpha u^{\alpha+1}\bar{u}^{\alpha}+\sum_{\beta=0}^{k}p_\beta(t)u^{r-\beta}\bar{u}^{\beta}+\cdots,$$

where $r>2k+1$, and the dots denote terms of degree higher than r. Make in system (2) the substitution

$$v=u+\sum_{\beta=0}^{r}a_\beta(t)u^{r-\beta}\bar{u}^{\beta}+\cdots,\qquad \bar{v}=\bar{u}+\sum_{\beta=0}^{r}\bar{a}_\beta(t)u^{\beta}\bar{u}^{r-\beta}+\cdots. \tag{3}$$

By elementary calculations we find that, in the variables $v,\bar{v}$, system (2) has the form

$$\dot{v}=-i\omega v+\sum_{\alpha=1}^{k}C_\alpha v^{\alpha+1}\bar{v}^{\alpha}+\sum_{\beta=0}^{r}[\dot{a}_\beta(t)-i\omega(r-2\beta-1)a_\beta(t)+p_\beta(t)]v^{r-\beta}\bar{v}^{\beta}+\cdots,$$

$$\dot{\bar{v}}=i\omega\bar{v}+\sum_{\alpha=1}^{k}\bar{C}_\alpha v^{\alpha}\bar{v}^{\alpha+1}+\sum_{\beta=0}^{r}[\dot{\bar{a}}_\beta(t)+i\omega(r-2\beta-1)\bar{a}_\beta(t)+\bar{p}_\beta(t)]v^{\beta}\bar{v}^{r-\beta}+\cdots, \tag{4}$$

where the dots still denote terms of degree higher than r.

Let us now determine what conditions the frequency ω must satisfy in order that, with the aid of the substitution (3), it be possible to maximally simpli-

the terms of degree r in the right-hand side of system (4). Setting

$$a_\beta(t)=\sum_{\gamma=-\infty}^{\infty}a_{\beta,\gamma}e^{i\gamma t},\qquad p_\beta(t)=\sum_{\gamma=-\infty}^{\infty}p_{\beta,\gamma}e^{i\gamma t},$$

we find that, in order to reduce to zero all terms of degree r, it is necessary that the $a_{\beta,\gamma}$ satisfy equations of the form

$$[i\gamma-i\omega(r-2\beta-1)]a_{\beta,\gamma}+p_{\beta,\gamma}=0. \tag{5}$$

In the case when r is even, we may assume that $\omega(r-2\beta-1)$ is not an integer for $\beta=0,1,\ldots,r$, and, consequently,

Fig. 2

Figure 2: Fig. 2

Fig. 1

Fig. 2

in this case system (5) has a unique solution. Using obvious estimates for $p_{\beta,\gamma}$, it is not difficult to establish that the function $a_\beta(t)$ will be analytic in the strip $|\operatorname{Im} t| < \delta_0$. Thus, in the case considered we can reduce to zero all terms of degree r. In the case when r is odd, the coefficient of $a_{\frac{r-1}{2},0}$, obviously, vanishes independently of ω. However, assuming that $\omega(r-2\beta-1)$ is not an integer for $\beta = 0, 1, \dots, \frac{r-3}{2}, \frac{r+1}{2}, \dots, r$, we can reduce to zero all terms of degree r, except the term

$$p_{\frac{r-1}{2},0}\, v^{\frac{r+1}{2}}\, \bar{v}^{\frac{r-1}{2}}.$$

In what follows, the form to which the terms of degree r have been reduced will be called canonical.

2. Conditions for strong resonance. Let now $\omega = m/n + \Delta\omega$, where $n \geqslant 3$, $|\Delta\omega| < 1/n(n-1)$, and m and n are relatively prime integers. Then, applying the transformation (3) a sufficient number of times, we bring system (1) to the form

$$\dot{u} = -i\left(\frac{m}{n} + \Delta\omega\right)u + \sum_{\alpha=1}^{\left[\frac{n-3}{2}\right]} C_\alpha(\Delta\omega) u^{\alpha+1}\bar{u}^\alpha + \sum_{\beta=0}^{n-1} p_\beta(t) u^{n-1-\beta}\bar{u}^\beta + \cdots, \quad (6)$$

$$\dot{\bar{u}} = i\left(\frac{m}{n} + \Delta\omega\right)\bar{u} + \sum_{\alpha=1}^{\left[\frac{n-3}{2}\right]} \overline{C}_\alpha(\Delta\omega) u^{\alpha}\bar{u}^{\alpha+1} + \sum_{\beta=0}^{n-1} \bar{p}_\beta(t) u^{\beta}\bar{u}^{n-1-\beta} + \cdots.$$

Let us now see to what form, by means of the transformation (3), one can reduce the terms of degree $(n-1)$ in the right-hand side of system (6). From equation (5) it follows that, putting in the transformation (3)

$$a_{\beta,\gamma} = \frac{ip_{\beta,\gamma}}{\gamma - (m/n + \Delta\omega)(n - 2\beta - 2)}$$

for those values of the indices β, γ for which

$$\gamma - \frac{m}{n}(n - 2\beta - 2) \neq 0,$$

we shall, by means of the transformation chosen in this way, reduce system (6) to the form

$$\dot{v} = -i\left(\frac{m}{n} + \Delta\omega\right) v + \sum_{\alpha=1}^{\left[\frac{n-3}{2}\right]} C_\alpha(\Delta\omega) v^{\alpha+1}\bar{v}^\alpha + p_{n-1,-m} e^{-imt}\bar{v}^{n-1} + \cdots,$$

$$\dot{\bar{v}} = i\left(\frac{m}{n} + \Delta\omega\right) \bar{v} + \sum_{\alpha=1}^{\left[\frac{n-3}{2}\right]} \bar{C}_\alpha(\Delta\omega) v^{\alpha}\bar{v}^{\alpha+1} + \bar{p}_{n-1,-m} e^{imt} v^{n-1} + \cdots, \qquad (7)$$

if n is odd, and to the form

$$\dot{v} = -i\left(\frac{m}{n} + \Delta\omega\right) v + \sum_{\alpha=1}^{\left[\frac{n-3}{2}\right]} C_\alpha(\Delta\omega) v^{\alpha+1}\bar{v}^\alpha + p_{n-1,-m} e^{-imt}\bar{v}^{n-1} +$$

$$+ p_{\frac{n}{2}-1,0}\, v^{n/2}\bar{v}^{n/2-1} + \cdots, \qquad (7')$$

$$\dot{\bar{v}} = i\left(\frac{m}{n} + \Delta\omega\right) \bar{v} + \sum_{\alpha=1}^{\left[\frac{n-3}{2}\right]} \bar{C}_\alpha(\Delta\omega) v^{\alpha}\bar{v}^{\alpha+1} + \bar{p}_{n-1,-m} e^{imt} v^{n-1} +$$

$$+ \bar{p}_{\frac{n}{2}-1,0}\, v^{n/2-1}\bar{v}^{n/2} + \cdots,$$

if n is even.

Next, putting in systems (7) and (7′)

$$v = \rho e^{-i\left(\frac{m}{n}t - \varphi\right)}, \qquad \bar{v} = \rho e^{i\left(\frac{m}{n}t - \varphi\right)},$$

we obtain that in the variables ρ, φ both systems have the form

$$\dot{\rho} = \sum_{\alpha=1}^{\left[\frac{n-3}{2}\right]} A_\alpha(\Delta\omega)\rho^{2\alpha+1} + (a_0 + a_1\cos n\varphi + b_1 \sin n\varphi)\rho^{n-1} + \cdots, \qquad (8)$$

$$\dot{\varphi} = -\Delta\omega + \sum_{\alpha=1}^{\left[\frac{n-3}{2}\right]} B_\alpha(\Delta\omega)\rho^{2\alpha} + (b_0 - a_1\sin n\varphi + b_1 \cos n\varphi)\rho^{n-2} + \cdots,$$

where $a_0 = b_0 = 0$ if n is odd.

Suppose now that for $\Delta\omega = 0$

Fig. 3

Figure 3: Fig. 3

$$A_\alpha(\Delta\omega) = B_\alpha(\Delta\omega) = 0 \quad \left(\alpha = 1, \dots, \left[\frac{n-3}{2}\right]\right), \qquad a_0^2 + b_0^2 < a_1^2 + b_1^2. \quad (9)$$

In this case, for $\Delta\omega = 0$ the origin will be an equilibrium position of saddle type. The behavior of the trajectories of system (8) in this case is well known (see [(3)], Chap. II). Conditions (9) will be called the conditions of strong resonance.

3. Behavior of the trajectories of system (1) in the case of strong resonance. Let us now see what the behavior of the trajectories of system (8) is for small $\Delta\omega \neq 0$, in the case when conditions (9) are satisfied. For this purpose we make in system (8) the substitution $\rho = \mu u$, $\varphi = v$ and $t = \mu^{2-n}\tau$. After the substitution we obtain:

$$\frac{du}{d\tau} = (a_0 + a_1 \cos nv + b_1 \sin nv)u^{n-1} + \mu F(u, v, \mu^{2-n}\tau, \mu),$$

$$\frac{dv}{d\tau} = \pm 1 + (b_0 - a_1 \sin nv + b_1 \cos nv)u^{n-2} + \mu G(u, v, \mu^{2-n}\tau, \mu), \qquad (10)$$

where $\mu = \sqrt[n-2]{|\Delta\omega|}$, while the functions $F(u, v, \theta, \mu)$ and $G(u, v, \theta, \mu)$ are analytic in u, v, θ, and μ, periodic in v with period 2π and in θ with period $2n\pi$. We must, obviously, determine the location of the separatrix of system (10)

for small $\mu \neq 0$, i.e., for small $\Delta\omega \neq 0$. To solve this problem we shall use the results of note (4). It follows from these results that, for $a_0 \neq 0$, the arrangement of the separatrix of system (10) will be rough, i.e., independent of the choice of the functions $F(u, v, \theta, \mu)$ and $G(u, v, \theta, \mu)$; its form is shown in Fig. 1 if $a_0 < 0$, and in Fig. 2 if $a_0 > 0$. In the case $a_0 = 0$, the arrangement of the separatrix will depend on the particular choice of the functions $F(u, v, \theta, \mu)$ and $G(u, v, \theta, \mu)$ and may have any of the forms shown in Figs. 1–4. In these figures the case $\Delta\omega > 0$ is shown. The case $\Delta\omega < 0$ is obtained by mirror reflection. The arrows in the figures show in which direction the points of the corresponding branches of the separatrix are displaced under a time shift by $2n\pi$.

Fig. 3

It is necessary to note that in the case when system (1) possesses an integral invariant, the arrangement of the separatrix of system (10) can have only the form shown in Figs. 3–4; moreover, the arrangement indicated in Fig. 4 will be more typical in the sense that, in order for the arrangement of the separatrices to have the form shown in Fig. 3, the functions $f(x, y, t)$ and $g(x, y, t)$ entering

Fig. 4

Figure 4: Fig. 4

into system (1) must satisfy some infinite number of functional conditions. On the other hand, in this case, as was already noted in note (1), the length of the segment CD, which characterizes the maximum deviation of the corresponding branches of the separatrix from one another, is of order $e^{-\alpha/|\Delta\omega|}$, where $\alpha > 0$. In investigating some problems this phenomenon may be neglected and one may assume that $|CD| = 0$, but only in those cases when $e^{-\alpha/|\Delta\omega|} \ll 1$, i.e., when $\Delta\omega$ is sufficiently small. For example, in studying passage through resonance, such a simplification can be made only when the velocity of passage through resonance v satisfies the inequality

$$v \gg |\Delta\omega| e^{-\alpha/|\Delta\omega|},$$

where $|\Delta\omega|$ is the width of the resonance.

Fig. 4

In conclusion I take this opportunity to express my gratitude to S. V. Fomin for his constant interest in my work.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.