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Abstract

Full Text

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UNIVALENT MAPPINGS OF PLANE DOMAINS AND SETS OF PRIME ENDS OF A DOMAIN OF GENERALIZED MEASURE ZERO

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The boundary properties of analytic functions defined in arbitrary simply connected domains remain unstudied. It is natural to begin this study by finding necessary and sufficient conditions for the set of prime ends of an arbitrary simply connected domain, under its conformal mapping onto a disk, to pass into a set of points of the boundary circle of Lebesgue measure zero; moreover, these conditions must be expressed in metric terms referring to the domain itself (and not to the mapping functions).

The results of the present note do not contain a complete solution of the problem posed, but they pertain to the same question. In addition, we consider mappings more general than conformal ones. For univalent Q-quasiconformal mappings of domains with rectifiable boundaries onto one another, a set of boundary points of linear (Lebesgue) measure zero may, as is known, pass into a set of positive measure. However, our result (Theorem 4) shows that it is possible to single out a broad class of perfect sets of measure zero which always pass into such sets also under Q-quasiconformal mappings and even under more general mappings.

1°. Consider an arbitrary simply connected domain A in the plane of the complex variable z, containing a fixed point O. In this plane we consider the spherical metric (R), obtained by stereographic projection onto this plane of the Riemann sphere of radius R, tangent to the plane at the point O. We shall consider the topological space

$$\widetilde{A} = A \cup \Gamma_A,$$

where $\Gamma_A = \Gamma_A(e_\alpha)$ is the set of prime ends e_α (in the sense of C. Carathéodory) with the natural topology (see, for example, ([1]), p. 76). The set Γ_A, considered as a subspace of $\widetilde{A}$, is a bicompact space. This set, as is known, is cyclically ordered; therefore one may speak of intervals and segments of this set. One may also consider perfect sets of prime ends defined in the usual way—subsets of Γ_A.

Definition 1. Let $e = e(e_\alpha)$ be some set of prime ends e_α of the domain A. If for every $\varepsilon > 0$ there exists a finite or countable set of cross-cuts $\{a_n\}$ of the domain A, separating the set e from the point O, and with the sum of the spherical lengths (in the spherical metric (R))

$$\sum_n L_R(a_n) < \varepsilon,$$

then the set e will be called a **set of prime ends of generalized measure zero**, or, briefly, an N_R**-set**.

Definition 2. Let e be an arbitrary set of prime ends of the domain A. The **outer measure** $m_R^* e$ of the set e is the number

$$m_R^* e = \inf \left\{ \sum_n L_R(a_n) \right\},$$

where the lower bound is taken over all systems $\{a_n\}$ separating e from the point O in A.

The N_R-sets of prime ends defined above possess the usual properties of sets of measure zero, while the concept of outer measure has the usual properties of an outer measure defined abstractly. It is also obvious that every set of prime ends has finite outer measure (in the spherical metric), and that a set e is an N_R-set if and only if $m_R^* e = 0$.

Theorem 1. *If the domain A has a rectifiable (in the metric (R)) boundary Γ_A, then the set of prime ends e of the domain A is an N_R-set if and only if the corresponding point set on the curve Γ_A has ordinary (Lebesgue in the spherical metric) linear measure zero.*

2°. Let A and B be domains having the properties described in 1°, situated respectively in the z- and w-planes. In the w-plane we shall consider the spherical metric (r) ($r \neq R$ is not excluded). Let $w = T(z)$ be an arbitrary homeomorphism between the topological spaces $\widetilde{A}$ and $\widetilde{B}$, $T(0) = 0$*.

Definition 3. We shall say that the homeomorphism T has the N-property on Γ_A if the set $T(e)$ is an N_r-set for every N_R-set e of prime ends of the domain A.

Theorem 2 (an analogue of N. N. Luzin's theorem ([4])). *If the homeomorphism $w = T(z)$, $T(\widetilde{A}) = B$, does not have the N-property on Γ_A, then there always exists a perfect N_R-set $\pi = \pi(e_\alpha)$ of prime ends of the domain A such that $m_r^* T(\pi) > 0$**.*

3°. We now give a theorem providing a sufficient condition for, under the homeomorphism T, a perfect N_R-set of prime ends of the domain A to pass into an N_r-set of prime ends of the domain A. In doing so, one has to impose restrictions both on the class of mappings and on the N_R-sets.

Theorem 3. *Let $w = T(z) = f_1(x, y) + i f_2(x, y)$, $z = x + iy$, $T(0) = 0$, be a topological mapping of the domain A onto the domain B, where f_1, f_2 have continuous first partial derivatives with respect to x and y, and*

$$\iint_A \frac{\sum_{j=1}^2 \operatorname{grad}^2 f_j \, dx \, dy}{\left[1 + \sum_{j=1}^2 \left(\frac{f_j}{2r}\right)^2\right]^2} < +\infty$$

(integration is understood in the sense of Lebesgue). Denote by $\pi = \pi(e_\alpha)$ a perfect set of prime ends of the domain with the following property: there exists a number $k < +\infty$ such that, for every $\varepsilon > 0$, the set π can be separated in the domain A from the point O by means of a system of cuts with total spherical (in the metric (R)) length $L_R(\varepsilon) < \varepsilon$ in such a way that the number of cuts in the system satisfies the inequality

$$n(\varepsilon) < k \ln \frac{1}{L_R(\varepsilon)}.$$

Under these conditions the set $T(\pi)$ of prime ends of the domain B will be an N_r-set.

4°. We formulate an example of a concretization of Theorem 3.

Theorem 4. *Let the conditions of Theorem 3 be fulfilled, as well as the additional condition that the boundary Γ_A is rectifiable (in the spherical metric (R)). Let π be a perfect point set of linear (Lebesgue in the spherical met-*

* T may be conformal, quasiconformal, or even a more general topological mapping, provided that T induces a boundary correspondence by prime ends as in the conformal case (see ([2],[3])).

** Moreover, as in the classical case, the values $T(z)$ on any portion of the set π also form a set of prime ends of positive outer measure.

of measure zero on the curve Γ_A, satisfying the condition: there exists a number k, $0 < k < \infty$, such that for all n

$$L_R(\gamma) - \sum_{j=1}^{n} L_R(\gamma_j) > e^{-k/n},$$

where γ is the smallest segment (in length in the spherical metric) on the curve Γ_A containing the set π, and γ_j are intervals on Γ_A adjacent to π; moreover, for each γ_{i_1} in the sequence $\{\gamma_j\}$ there are no more than n_0 intervals equal in length to the interval γ_{i_1} (n_0 does not depend on i_1).

Under these conditions the set $T(\pi)$ will be an N_r-set of prime ends of the domain B.

Remark 1. Under a conformal mapping of the disk onto a domain with non-rectifiable boundary, a perfect set of points of measure zero on the boundary of the disk may pass into a set of prime ends of the domain of positive outer measure. Here the example of N. N. Luzin and I. I. Privalov is suitable (([5], pp. 305, 308). The circumstance noted suggests the necessity of extending the concept of a set of prime ends of generalized measure zero so that all conformal mappings possess the N-property in arbitrary simply connected domains.

Remark 2. We used different spherical metrics in different planes for economy in writing the results. If a domain bounded in the ordinary sense is considered, then in the corresponding plane one may put R (or r) equal to ∞ and consider all lengths in the ordinary sense. If, however, an unbounded domain is considered, then one may put R (or r) equal to $1/2$ and consider lengths in the spherical sense.

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CITED LITERATURE

[1] G. D. Suvorov, *Matem. sborn.*, **33** (75), 1 (1953).
[2] G. D. Suvorov, *Matem. sborn.*, **45** (87), 2 (1958).
[3] G. D. Suvorov, *Sibirsk. matem. zhurn.*, 1, 3, 492 (1960).
[4] N. N. Luzin, *Sob. soch.*, I, USSR Academy of Sciences Press, 1953, pp. 123–126.
[5] I. I. Privalov, *Boundary Properties of Analytic Functions*, 1950.

Note: Figure translations are in progress. See original paper for figures.