# MATHEMATICS

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**Abstract**

**Full Text**

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# ON BIFURCATION POINTS OF A CERTAIN CLASS OF EQUATIONS

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In a complex Banach space $E$, consider the equation

$$\varphi = \lambda B\varphi + D(\varphi, \lambda), \qquad (1)$$

where $\lambda$ is a complex parameter; $B$ is a linear completely continuous operator; the operator $D(\varphi, \lambda)$ is analytic jointly in $\varphi, \lambda$ for $\varphi$ belonging to some neighborhood of zero in the space $E$ and $\lambda$ belonging to some open domain $M$ of the complex plane, and satisfies the condition

$$\lim_{\|\varphi\|\to 0} \|D(\varphi, \lambda)\| \, \|\varphi\|^{-1} = 0 \qquad (\lambda \in M).$$

In the present note, the law of linearization is proved for the problem of bifurcation points of equation (1)*.

**Theorem.** *In order that $\lambda_0$ $(\lambda_0 \in M)$ be a bifurcation point of equation (1), it is necessary and sufficient that $\lambda_0$ be a characteristic value of the operator $B$.*

The necessity of the condition of the theorem follows directly from the Hildebrandt–Graves theorem on the implicit function ([1]).

We give a brief exposition of the proof of sufficiency.

1°. Let $\lambda_0$ be a characteristic value of the operator $B$. Denote $\lambda - \lambda_0 = \mu$ and write equation (1) in the form

$$\varphi = (\lambda_0 + \mu)B\varphi + \sum_{k+j\geq 2}^{\infty} \mu^j D_{kj}\varphi^k, \qquad (2)$$

where $D_{kj}\varphi^k$ is a homogeneous form of degree $k$.

To investigate the question of small solutions of equation (2) for small values of the parameter $\mu$, we apply to this equation the Lyapunov–Schmidt method of transition to branching equations.

Let $e_1, \dots, e_n$ be a complete system of linearly independent eigenvectors of the operator $B$ corresponding to the eigenvalue $1/\lambda_0$, and let $q_1, \dots, q_n$ be such linear functionals that $(q_i, e_j) = \delta_{ij}$. Let $\psi_1, \dots, \psi_n$ be linearly independent eigenfunctionals of the adjoint operator $B^*$, corresponding to the same eigenvalue, and let $p_1, \dots, p_n$ be such elements of $E$ that $(\psi_i, p_j) = \delta_{ij}$. Then for each small solution $\varphi$ of equation (2), the numbers $c_i = (q_i, \varphi)$, $i = 1, \dots, n$, satisfy the branching equations

$$c_i - (q_i, f(c_1, \dots, c_n, \mu)) = 0 \qquad (i = 1, \dots, n), \tag{3}$$

where $f(c_1, \dots, c_n, \mu)$ is a small solution of the equation

$$\varphi = \lambda_0 B\varphi - \sum_{i=1}^{n} (q_i, \varphi) p_i + \mu B\varphi + \sum_{k+j\geq 2}^{\infty} \mu_{kj}^{jD} \varphi^k + \sum_{i=1}^{n} c_{ip} i. \tag{4}$$

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* The same result was obtained in ([4]) for the equation $\varphi = \lambda A\varphi$, where $A\varphi$ is analytic and completely continuous and $A0 = 0$.

Let us note that equation (4), for small values of the parameters $c_1, \dots, c_n, \mu$, has a unique small solution, and this solution can be represented in the form of a series in integral nonnegative powers of $c_1, \dots, c_n, \mu$.

2°. Consider system (3) for $\mu = 0$. It is known [(2)] that the functions $c_i - (q_i, f(c_1, \dots, c_n, 0))$, $i = 1, \dots, n$, are of order of smallness no lower than the second relative to $c_1, \dots, c_n$. Therefore, from the results of Cronin (see [(3)], pp. 177-180) it follows that, in the case when the zero solution of the system

$$c_i - (q_i, f(c_1, \dots, c_n, 0)) = 0 \qquad (i = 1, \dots, n) \tag{5}$$

is isolated, the index of this solution is greater than one (in the case of non-isolation of the zero solution of system (5), the zero solution of equation (2) for $\mu = 0$ is also non-isolated).

3°. Now consider system (3) for small $\mu$, different from zero, and show that the index of the zero solution of this system is equal to 1. Obviously, it is sufficient to prove that the system

$$c_i - (q_i, f(c_1, \dots, c_n, \mu)) = \alpha_i \qquad (i = 1, \dots, n) \tag{6}$$

for small $\alpha_1, \dots, \alpha_n$ has a unique small solution depending continuously on $\alpha_1, \dots, \alpha_n$. To prove the latter assertion, consider the equation

$$\varphi = (\lambda_0 + \mu) B\varphi + \sum_{k+j\geq 2}^{\infty} \mu^j D_{kj} \varphi^k + \sum_{i=1}^{n} \alpha_i p_i. \tag{7}$$

This equation is equivalent to system (6) in the sense that to every small solution $\overline{\varphi}$ of equation (7) there corresponds, by the formulas

$$\overline{c}_i = (q_i, \overline{\varphi}) + \alpha_i \qquad (i = 1, \dots, n),$$

a certain small solution $\overline{c}_1, \dots, \overline{c}_n$ of system (6), and conversely, if $\overline{c}_1, \dots, \overline{c}_n$ is a small solution of system (6), then

$$\overline{\varphi} = f(\overline{c}_1, \dots, \overline{c}_n, \mu)$$

is a small solution of equation (7).

Since equation (7) has a unique small solution and this solution depends analytically on $\alpha_1, \dots, \alpha_n$, system (6) also has a unique small solution depending analytically on $\alpha_1, \dots, \alpha_n$.

4°. From the discrepancy between the indices of the zero solution of system (3) for $\mu = 0$ and for small $\mu$ different from zero, it obviously follows that $\mu = 0$ is a bifurcation point of system (3). But then for equation (2) as well, $\mu = 0$ is a bifurcation point, as was required to prove.

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## CITED LITERATURE

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[4] V. B. Melamed, DAN, 145, No. 3, 531 (1962).

*Note: Figure translations are in progress. See original paper for figures.*