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1963

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Abstract

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INTEGRAL REPRESENTATION OF FUNCTIONS HOLOMORPHIC IN CONVEX DOMAINS OF THE SPACE C^n

(Presented by Academician V. I. Smirnov, 4 III 1963)

1. Cauchy' s integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, d\zeta \tag{1}$$

(where the function $f(z)$ is holomorphic in the domain D and continuous in the closed domain $\overline{D}$; ∂D consists of a finite number of closed rectifiable Jordan curves) plays a very important role in the study of holomorphic functions of one complex variable, which, in our opinion, is explained mainly by two properties of this formula: 1) formula (1) is universal, i.e. it is valid and has one and the same form for any D; 2) the Cauchy kernel $\frac{1}{\zeta - z}$ is holomorphic in z (this ensures the holomorphy of the Cauchy-type integral, etc.).

For holomorphic functions of several complex variables it is not possible to obtain an integral formula possessing the two indicated properties of Cauchy' s formula: a) the Martinelli–Bochner integral representation (see ([1]), § 21) is universal, but has a nonholomorphic kernel; b) the integral representations of Weil (see ([1]), § 22), Bergman ([2]), Hua Loo-Keng ([3]), Temlyakov ([4]), and others (see ([5,6]); ([1]), § 23) are not universal, but do possess holomorphic kernels.

In the present paper an integral formula (2) with a holomorphic kernel is obtained for convex domains of the space C^n of n complex variables $z_1, z_2, \dots, z_n$. We know two proofs of this formula: A) formula (2) can be derived from the Cauchy–Fantappiè integral representation indicated by Leray ([7]); B) formula (2) can be obtained elementarily, without relying on the Cauchy–Fantappiè formula. We shall give the second proof. We note that, as Norguet ([8]) showed, the Martinelli–Bochner and Weil integral representations can also be obtained from the Cauchy–Fantappiè formula.

2.

Let the domain

$$D = \{(z_1, z_2, \dots, z_n) : \Phi(z_1, \bar{z}_1, z_2, \bar{z}_2, \dots, \bar{z}_n, z_n) < 0\}$$

be convex and bounded, and let the function Φ be twice continuously differentiable and all first-order derivatives of Φ not vanish simultaneously at points of the boundary ∂D of the domain D.

Theorem. If the function $f(z_1, z_2, \dots, z_n)$ is holomorphic in the domain D and continuous in the closed domain $\overline{D}$, then for points $(z_1, z_2, \dots, z_n) \in D$

$$f(z_1, \dots, z_n) = \frac{(n-1)!}{(2\pi i)^n} \int_{\partial D} \frac{f(\zeta_1, \dots, \zeta_n) \left[\sum_{k=1}^{n} \delta_k \left(\bigwedge_{1 \le j \le n;\, j \ne k} d\bar{\zeta}_j\right)\right] \bigwedge_{1 \le j \le n} d\zeta_j}{\left[\Phi'_{\zeta_1} \cdot (\zeta_1 - z_1) + \dots + \Phi'_{\zeta_n} \cdot (\zeta_n - z_n)\right]^n}, \tag{2}$$

where $\bigwedge$ is the sign of exterior multiplication,

$$\delta_k = \begin{vmatrix} \Phi'_{\zeta_1} & \Phi'_{\zeta_2} & \dots & \Phi'_{\zeta_n} \\ \Phi''_{\zeta_1\bar{\zeta}_1} & \Phi''_{\zeta_2\bar{\zeta}_1} & \dots & \Phi''_{\zeta_n\bar{\zeta}_1} \\ \dots & \dots & \dots & \dots \\ \Phi''_{\zeta_1\bar{\zeta}_{k-1}} & \Phi''_{\zeta_2\bar{\zeta}_{k-1}} & \dots & \Phi''_{\zeta_n\bar{\zeta}_{k-1}} \\ \Phi''_{\zeta_1\bar{\zeta}_{k+1}} & \Phi''_{\zeta_2\bar{\zeta}_{k+1}} & \dots & \Phi''_{\zeta_n\bar{\zeta}_{k+1}} \\ \dots & \dots & \dots & \dots \\ \Phi''_{\zeta_1\bar{\zeta}_n} & \Phi''_{\zeta_2\bar{\zeta}_n} & \dots & \Phi''_{\zeta_n\bar{\zeta}_n} \end{vmatrix}.$$

Proof will be carried out, for simplicity, in the case of two complex variables; then formula (2) takes the form

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{\partial D} \frac{f(\zeta_1, \zeta_2) \left(\begin{vmatrix} \Phi'_{\zeta_1} & \Phi'_{\zeta_2} \\ \Phi'_{\bar{\zeta}_1} & \Phi'_{\zeta_2\bar{\zeta}_1} \end{vmatrix} d\bar{\zeta}_1 + \begin{vmatrix} \Phi'_{\zeta_1} & \Phi'_{\zeta_2} \\ \Phi'_{\bar{\zeta}_2} & \Phi'_{\zeta_2\bar{\zeta}_2} \end{vmatrix} d\bar{\zeta}_2 \right) \wedge d\zeta_1 \wedge d\zeta_2}{\left[\Phi'_{\zeta_1}(\zeta_1 - z_1) + \Phi'_{\zeta_2}(\zeta_2 - z_2)\right]^2}. \tag{3}$$

Let us first note that, by virtue of the conditions imposed by us on the function Φ, at every point $(\zeta_1, \zeta_2) \in \partial D$ there exists a tangent analytic plane

$$\{(z_1, z_2) : \Phi'_{\zeta_1}(\zeta_1 - z_1) + \Phi'_{\zeta_2}(\zeta_2 - z_2) = 0\}.$$

Since the domain D is convex, it follows that

$$\Phi'_{\zeta_1}(\zeta_1 - z_1) + \Phi'_{\zeta_2}(\zeta_2 - z_2) \ne 0$$

for $(z_1, z_2) \in D$.

I. First suppose that the function $f(z_1, z_2)$ is holomorphic in the closed domain $\overline{D}$. Represent the boundary ∂D of the domain D in the form $\partial D = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_1 = \{(\zeta_1, \zeta_2) : (\zeta_1, \zeta_2) \in \partial D,\ |\zeta_1 - z_1 - \zeta_2 + z_2| \geq \sigma\},$$

$$\Gamma_2 = \{(\zeta_1, \zeta_2) : (\zeta_1, \zeta_2) \in \partial D,\ |\zeta_1 - z_1 - \zeta_2 + z_2| < \sigma\}, \quad \sigma > 0.$$

Denote the right-hand side of formula (3) by I. Then $I = I_1 + I_2$, where the integrals I_1 and I_2 are obtained from I by replacing the set of integration ∂D respectively by Γ_1 and Γ_2. Obviously, for sufficiently small σ the inequality $|I_2| < \varepsilon/2$ holds, where ε is any fixed positive number.

Since the integrand in formula (3) is the differential of the exterior differential form

$$\mu = \frac{f(\zeta_1, \zeta_2)(\Phi'_{\zeta_1} + \Phi'_{\zeta_2})\, d\zeta_1 \wedge d\zeta_2}{(\zeta_1 - z_1 - \zeta_2 + z_2)\left[\Phi'_{\zeta_1}(\zeta_1 - z_1) + \Phi'_{\zeta_2}(\zeta_2 - z_2)\right]},$$

then, with the aid of Stokes' formula (see (9), § 6), we obtain

$$I_1 = \int_{\partial\Gamma_1} \mu = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_{C_{t,\sigma}} \frac{f(\zeta_1, \zeta_1 - z_1 + z_2 - \sigma e^{it})(\Phi'_{\zeta_1} + \Phi'_{\zeta_2})\, d\zeta_1}{\Phi'_{\zeta_1}(\zeta_1 - z_1) + \Phi'_{\zeta_2}(\zeta_2 - z_2)},$$

where

$$C_{t,\sigma} = \{(\zeta_1, \zeta_2) : (\zeta_1, \zeta_2) \in \partial D,\ \zeta_1 - z_1 - \zeta_2 + z_2 = \sigma e^{it}\},$$

$$\partial\Gamma_1 = \bigcup_{0 \leq t \leq 2\pi} C_{t,\sigma}.$$

Hence we find

$$I_1 = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_{C_{t,\sigma}} \frac{f(\zeta_1, \zeta_1 - z_1 + z_2 - \sigma e^{it})\, d\zeta_1}{\zeta_1 - z_1} +$$

$$+ \frac{\sigma}{4\pi^2 i} \int_0^{2\pi} dt \int_{C_{t,\sigma}} \frac{f(\zeta_1, \zeta_1 - z_1 + z_2 - \sigma e^{it})\Phi'_{\zeta_2} e^{it}\, d\zeta_1}{(\zeta_1 - z_1)\left[\Phi'_{\zeta_1}(\zeta_1 - z_1) + \Phi'_{\zeta_2}(\zeta_2 - z_2)\right]} = I_3 + \sigma I_4.$$

The point (z_1, z_2) is a fixed point of the domain D; therefore there exists an $h_1 > 0$ such that

$$|\Phi'_{\zeta_1}(\zeta_1 - z_1) + \Phi'_{\zeta_2}(\zeta_2 - z_2)| > h_1$$

for $(\zeta_1, \zeta_2) \in \partial\Gamma_1 \subset \partial D$. We further note that

$$\partial\Gamma_1 = \{(\zeta_1, \zeta_2) : (\zeta_1, \zeta_2) \in \partial D,\ |\zeta_1 - z_1 - \zeta_2 + z_2| = \sigma\},$$

for sufficiently small σ, does not intersect the plane

$$\{(\zeta_1, \zeta_2) : \zeta_1 - z_1 = 0\}.$$

Indeed, otherwise the intersection

$$\partial D \cap \{(\zeta_1, \zeta_2) : \zeta_1 - z_1 - \zeta_2 + z_2 = 0\} \cap \{(\zeta_1, \zeta_2) : \zeta_1 - z_1 = 0\}$$

would not be empty, and we arrive at a contradiction, since the only common point of the two analytic planes

$$\{(\zeta_1, \zeta_2) : \zeta_1 - z_1 - \zeta_2 + z_2 = 0\} \quad \text{and} \quad \{(\zeta_1, \zeta_2) : \zeta_1 - z_1 = 0\}$$

is the point $(z_1, z_2) \notin \partial D$. From the compactness of the set $\partial\Gamma_1$ it follows that there exists an $h_2 > 0$ such that

$$|\zeta_1 - z_1| > h_2$$

for $(\zeta_1, \zeta_2) \in \partial\Gamma_1$. Therefore the quantity I_4 is bounded in modulus. But then, for sufficiently small σ,

$$\sigma|I_4| < \varepsilon/2.$$

Further,

$$I_3 = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_{C^+_{t,\sigma}} \frac{f(\zeta_1, \zeta_1 - z_1 + z_2 - \sigma e^{it})\, d\zeta_1}{\zeta_1 - z_1} =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_1, z_2 - \sigma e^{it})\, dt = f(z_1, z_2).$$

Thus,

$$I = I_1 + I_2 = I_3 + \sigma I_4 + I_2 = f(z_1, z_2) + (\sigma I_4 + I_2),$$

where

$$|\sigma I_4 + I_2| \leq \sigma|I_4| + |I_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, by the arbitrariness of ε, it follows that $I = f(z_1, z_2)$.

II. Now suppose that the function $f(z_1, z_2)$ is holomorphic in the domain D and continuous in the closed domain $\overline{D}$. Then $f(z_1, z_2)$ can be represented as the limit of the sequence

$$\{f_m(z_1, z_2) \equiv f(z_1(1 - 1/m),\, z_2(1 - 1/m)), \quad m = 2, 3, ...\},$$

consisting of functions holomorphic in the closed domain $\overline{D}$ and converging uniformly in this closed domain (we assume that the point $(0, 0) \in D$; otherwise one can make a parallel translation). Writing formula (3) for each of the functions $f_m(z_1, z_2)$ and passing to the limit as $m \to \infty$ in both parts of the equality obtained, we arrive at formula (3) for the function $f(z_1, z_2)$.

3. We note some consequences of the theorem proved.

A. In the case $n = 1$, formula (2) yields the Cauchy formula (1).

B. Let the bicircular domain

$$D = \{(z_1, z_2) : |z_2| < \Phi(|z_1|),\ 0 \le |z_1| \le r\}$$

be convex and bounded, and let the function Φ be twice continuously differentiable. Then for $(z_1, z_2) \in D$ the formula

$$f(z_1, z_2) = \frac{-1}{(2\pi i)^2} \int_{\partial D} \frac{f(\zeta_1, \zeta_2)\, d\varphi_1(|\zeta_1|) \wedge \dfrac{d\zeta_1}{\zeta_1} \wedge \dfrac{d\zeta_2}{\zeta_2}}{\left[1 - z_1\bar{\zeta}_1\, \dfrac{\varphi_1(|\zeta_1|)}{|\zeta_1|^2} - z_2\bar{\zeta}_2\, \dfrac{\varphi_2(|\zeta_2|)}{|\zeta_2|^2}\right]^2}, \qquad (4)$$

is valid, where

$$\varphi_1(|\zeta_1|) = \frac{|\zeta_1|\Phi'(|\zeta_1|)}{|\zeta_1|\Phi'(|\zeta_1|) - \Phi(|\zeta_1|)}, \quad \varphi_2(|\zeta_2|) = \varphi_2(\Phi(|\zeta_1|)) = 1 - \varphi_1(|\zeta_1|).$$

(4) is Temlyakov' s integral representation, written in another form (see ([4],[6]); ([4]), § 23). Formula (4) can also be obtained under more general assumptions regarding the smoothness of the function Φ (see ([6],[10])).

C. For the hyperellipsoid

$$D = \Big\{(z_1, z_2) : z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2,\ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{x_2^2}{c^2} + \frac{y_2^2}{d^2} < 1\Big\}$$

the integral representation (2) has the form

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{\partial D} \frac{f(\zeta_1, \zeta_2)(-b_1\beta\, d\bar{\zeta}_1 + b_2\alpha\, d\bar{\zeta}_2) \wedge d\zeta_1 \wedge d\zeta_2}{[\alpha(\zeta_1 - z_1) + \beta(\zeta_2 - z_2)]^2},$$

where

$$\alpha = a_1\zeta_1 + b_1\bar{\zeta}_1, \quad \beta = a_2\zeta_2 + b_2\bar{\zeta}_2, \quad a_1 = \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right), \quad b_1 = \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

$$a_2 = \frac{1}{2}\left(\frac{1}{c^2} - \frac{1}{d^2}\right), \quad b_2 = \frac{1}{2}\left(\frac{1}{c^2} + \frac{1}{d^2}\right).$$

4. One may consider an "integral of type (2)," i.e., the integral from the right-hand side of formula (2) of an arbitrary function $f(\zeta_1, \zeta_2)$, summable on the boundary ∂D of the domain D. This integral defines a function holomorphic in the domain D. In particular, integrals of Temlyakov type were studied in detail by us earlier ([11],[12]). It follows from these studies that the integral (2) for $n > 1$, generally speaking, is not equal to zero outside the domain D, while an integral of type (2) for $n > 1$, generally speaking, defines a function that is not holomorphic outside the domain D.

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Received
2 III 1963

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Note: Figure translations are in progress. See original paper for figures.