MATHEMATICS

B. Efimov

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

B. Efimov

ON DYADIC SPACES

(Presented by Academician P. S. Aleksandrov on 4 III 1963)

§ 1. Main results

In this note we use the concepts which we used in the note ([1]). Thus, D^τ, $H_w^{i(w)}$, $\chi(F, X)$ denote, respectively: the generalized Cantor discontinuum, its layer with base w, and the character of a neighborhood of the closed set F lying in X. As is known, a completely regular space X is called **dyadic** if it has at least one dyadic bicompact extension.

Theorem 1. If at every point x of a dyadic space X we have $\chi(x, X) \leq \mathfrak{m}$, then in the space X there is an everywhere dense set M of cardinality $\leq \mathfrak{m}$.

We shall say that a space X satisfies condition $(\mathfrak{m})$ if it contains a dense subset M of cardinality not exceeding $\mathfrak{m}$, and at each point $x \in M$ we have $\chi(x, X) \leq \mathfrak{m}$.

Theorem 2. A dyadic bicompactum R satisfying condition $(\mathfrak{m})$ has weight not exceeding $\mathfrak{m}$.

The following theorem follows from Theorems 1 and 2.

Theorem 3. The weight of a dyadic space X is equal to the least upper bound of the values $\chi(x, X)$, where $x \in M$, and M is any dense subset of X.

Corollary 1. A dyadic space X satisfying the first axiom of countability at the points of some everywhere dense subset of X is metrizable.

Corollary 2. A dyadic bicompact extension of a metric space is metrizable.

Hence:

Theorem 4. A bicompact extension rX of a metric space X is metrizable if and only if rX is dyadic.

§ 2. Auxiliary propositions

We shall need below the following easily proved propositions:

1) Through each fixed point $x \in D^\tau$ there passes a unique layer with the given base.

2) Two layers with one and the same base are either disjoint or identical.

3) If the layer $H_{w_1}^{i(w_1)}$ contains the layer $H_{w_2}^{i(w_2)}$, then $w_1 \subset w_2$.

4) If $w_1 \subset w_2$, then every layer with base w_1 can be represented in the form

$$H_{w_1} = \bigcup_{\xi} H_{w_2}^{\xi}.$$

5) If

$$H_{w_1}^{i(w_1)} \cap H_{w_2}^{j(w_2)} = \Lambda,$$

then $w_1 \cap w_2 \neq \Lambda$, and there is an index $\alpha \in w_1 \cap w_2$ such that $i(\alpha) \in i(w_1)$ is opposite to $j(\alpha) \in i(w_2)$.

6) $\chi(H_w^{i(w)}, D^\tau) = \operatorname{card} w$.

We shall call a pair (H_w, H_λ), consisting of a layer $H_w^{i(w)}$ and an elementary neighborhood

$$H_\lambda = H_{\alpha_1 \ldots \alpha_s}^{i_1 \ldots i_s},$$

connected if $H_w \cap H_\lambda \neq \Lambda$. A completely ordered sequence (1)

$$(H_{w_1}, H_{\lambda_1}), \ldots, (H_{w_\xi}, H_{\lambda_\xi}), \ldots, \quad \xi < \omega(\mathfrak{m}), \tag{1}$$

consisting of related pairs $(H_{w_\xi}, H_{\lambda_\xi})$, will be called **exact** if for every $\eta < \omega(\mathfrak{m})$ we have $H_{\lambda_\eta} \cap H_{w_\xi} = \Lambda$ for all $\xi < \eta$. Note that every subsequence of an exact sequence (1) is exact, and also that all elementary neighborhoods H_{λ_ξ}, $\xi < \omega(\mathfrak{m})$, occurring in (1), are distinct.

Main Lemma 1. If $F = \left[\bigcup_{\xi \in \Xi} H_{w_\xi}\right]$, where $\operatorname{card} w_\xi \leq \mathfrak{m}$ for all $\xi \in \Xi$, then there exists a set $\Xi_0 \subseteq \Xi$ of cardinality not exceeding $\mathfrak{m}$ such that $F = \left[\bigcup_{\xi \in \Xi_0} H_{w_\xi}\right]$.

Proof. Let $\mathfrak{m} = \aleph_\rho$; then $\mathfrak{n} = \aleph_{\rho+1}$ is an uncountable regular cardinal. Arguing by contradiction, we shall carry out the proof in two stages. In the first stage we construct an exact sequence of the form (1), ordered according to the type $\omega(\mathfrak{n})$, with $\operatorname{card} w_\xi \leq \mathfrak{m}$ for every $\xi < \omega(\mathfrak{n})$. In the second stage we show that the existence of such a sequence leads to a contradiction.

1st stage. Put $\left[\bigcup_{\xi \in \Xi} H_{w_\xi}\right] = P$. We shall carry out the construction by transfinite induction. For H_{w_1} take any layer from P, and for H_{λ_1} any elementary neighborhood of nonzero rank intersecting it. Suppose that for some $\zeta < \omega(\mathfrak{n})$ an exact sequence (2) has been constructed, where $H_{w_\xi} \in P$ if $\xi < \zeta$,

$$(H_{w_1}, H_{\lambda_1}), \ldots, (H_{w_\xi}, H_{\lambda_\xi}), \ldots, \quad \xi < \zeta. \tag{2}$$

Consider $B_\zeta = \left[\bigcup_{\xi<\zeta} H_{w_\xi}\right]$. Since the cardinality of the ordinal numbers less than ζ does not exceed $\mathfrak{m}$, it follows from the assumption that $B_\zeta \neq F$; consequently, $B_\zeta \subset F$, or $F \setminus B_\zeta = C_\zeta \neq \Lambda$. Since the closure of P is F, there exists a layer $H_{w_\zeta} \in P$ such that $H_{w_\zeta} \cap C_\zeta \neq \Lambda$. For some point $x \in H_{w_\zeta} \cap C_\zeta$ we find an elementary neighborhood $H_{\lambda_\zeta}(x) \subset D^\tau \setminus B_\zeta$. Thus $H_{\lambda_\zeta} \cap H_{w_\xi} = \Lambda$ for all $\xi < \zeta$. The pair $(H_{w_\zeta}, H_{\lambda_\zeta})$ is the desired one. Thus the exact sequence (3)

$$(H_{w_1}, H_{\lambda_1}), \dots, (H_{w_\xi}, H_{\lambda_\xi}), \dots, \quad \xi < \omega(\mathfrak{n}), \tag{3}$$

ordered according to the type $\omega(\mathfrak{n})$, has been constructed. Moreover, since all $H_{w_\xi} \in P$, we have $\operatorname{card} w_\xi \leq \mathfrak{m}$ for all $\xi < \omega(\mathfrak{n})$.

2nd stage. We show that the sequence (3) cannot exist in D^τ. By virtue of the uncountability and regularity of $\mathfrak{n}$, pass to an exact subsequence (4), cofinal in (3):

$$(H_{w_1^0}, H_{\lambda_1^0}), \dots, (H_{w_\xi^0}, H_{\lambda_\xi^0}), \dots, \quad \xi < \omega(\mathfrak{n}), \tag{4}$$

all the $H_{\lambda_\xi^0}$ of which have one and the same rank s. Further, since $H_{w_1^0} \cap H_{\lambda_\xi^0} = \Lambda$ for all $\xi > 1$, by virtue of 5), there is an index $\alpha \in w_1^0 \cap \lambda_\xi^0$ such that $i(\alpha) \in i(w_1^0)$ is opposite to $j(\alpha) \in i(\lambda_\xi^0)$. To each $a_\mu^0 \in w_1^0$ assign the set of those $H_{\lambda_\xi^0}$ whose bases contain the index a_μ^0. Since $\operatorname{card} w_1 \leq \mathfrak{m}$, while $\operatorname{card}\{H_{\lambda_\xi^0}\} = \mathfrak{n} > \mathfrak{m}$, there exists an exact subsequence (5), cofinal in (4):

$$(H_{w_1^1}, H_{\lambda_1^1}), \dots, (H_{w_\xi^1}, H_{\lambda_\xi^1}), \dots, \quad \xi < \omega(\mathfrak{n}), \tag{5}$$

all bases of whose neighborhoods contain one and the same index α_1, and the α_1-th coordinate is one and the same. Suppose that for some integer k we have constructed an exact sequence

$$(H_{w_1^k}, H_{\lambda_1^k}), \dots, (H_{w_\xi^k}, H_{\lambda_\xi^k}), \dots, \quad \xi < \omega(\mathfrak{n}), \tag{6}$$

cofinal with (5), and all bases of the neighborhoods (6) contain the same indices $\alpha_1, \dots, \alpha_k$, and the values $i(\alpha_1), \dots, i(\alpha_k)$ are the same for all neighborhoods (6).

Consider the pair $(H_{w_1^k}, H_{\lambda_1^k})$. Since $H_{w_1^k} \cap H_{\lambda_1^k} \neq \Lambda$, then, logically negating 5), either a) $w_1^k \cap \lambda_1^k = \Lambda$, or b) $e = w_1^k \cap \lambda_1^k \neq \Lambda$ and for all $\alpha \in e$ we have $i(\alpha) \in i(w_1^k)$ equal to $j(\alpha) \in i(\lambda_1^k)$. On the other hand, by virtue of the precision of (6), we have $H_{w_1^k} \cap H_{\lambda_\xi^k} = \Lambda$ for all $\xi > 1$. This means that $\lambda_\xi^k \cap w_1^k \neq \Lambda$, if $\xi > 1$. In case a), every λ_ξ^k intersects w_1^k in indices not belonging to $\alpha_1, \dots, \alpha_k$. But since $\operatorname{card}(w_1^k \setminus \alpha_1, \dots, \alpha_k) \leq \mathfrak{m}$, and $\operatorname{card}\{H_{\lambda_\xi^k}\} > \mathfrak{n}$, then, by the regularity of $\mathfrak{n}$, there exists a precise sequence (7), cofinal with (6):

$$(H_{w_1^{k+1}}, H_{\lambda_1^{k+1}}), \dots, (H_{w_\xi^{k+1}}, H_{\lambda_\xi^{k+1}}), \dots, \quad \xi < \omega(\mathfrak{n}), \tag{7}$$

all bases of whose neighborhoods contain one and the same index α_{k+1}, not belonging to $\alpha_1, \dots, \alpha_k$, and $i(\alpha_{k+1})$ is the same for them. In case b) we again find that every λ_ξ^k, $\xi > 1$, by virtue of 5), intersects w_1^k in indices not belonging to $\alpha_1, \dots, \alpha_k$. Thus in both cases we obtain a precise sequence (7), ordered according to type $\omega(\mathfrak{n})$, all bases of whose neighborhoods contain the same indices $\alpha_1, \dots, \alpha_{k+1}$, and the values $i(\alpha_1), \dots, i(\alpha_{k+1})$ are the same for them. Thus, in no more than s steps, we obtain the precise sequence (8)

$$(H_{w_1^s}, H_{\lambda_1^s}), \dots, (H_{w_\xi^s}, H_{\lambda_\xi^s}), \dots, \quad \xi < \omega(\mathfrak{n}), \tag{8}$$

all neighborhoods of which coincide, which contradicts the precision of (8). The lemma is proved.

Using the properties of the layers 1)–4), one can prove the following lemma:

Lemma 2. If $H = \left[\bigcup_{\xi\in\Xi_0} H_w^\xi\right]$, where w is the same for all $\xi \in \Xi_0$, then $H = \bigcup_{\xi\in\Xi_0} H_w^\xi$ and $\Xi_1 \supseteq \Xi_0$.

§ 3. **Proof of the principal theorems.**

Proof of Theorem 1. Let bX be the dyadic bicompact extension of X, and let f be a continuous mapping of D^τ onto bX. It can be shown that $\chi(x, bX) = \chi(x, X)$. Let $F_x = f^{-1}(x)$; then $\chi(F_x, D^\tau) \le \mathfrak{m}$, if $x \in X$. Using Proposition A from ([1]), we obtain that $F_x = \bigcup_{\xi\in\Xi_x} H_{w_x}^\xi$, where $\operatorname{card} w_x \le \mathfrak{m}$, if $x \in X$. Put $F = [f^{-1}X]$. Then

$$F = \left[\bigcup_{x\in X} F_x\right] = \left[\bigcup_{x\in X}\bigcup_{\xi\in\Xi_x} H_{w_x}^\xi\right] = \left[\bigcup_{\xi\in\Xi} H_{w_\xi}\right], \quad \text{if } \Xi = \bigcup_{x\in X} \Xi_x,$$

where $\operatorname{card} w(\xi) \le \mathfrak{m}$. Applying the basic Lemma 1, we obtain that there exists $\Xi_0 \subset \Xi$ such that $\operatorname{card}\Xi_0 \le \mathfrak{m}$ and $\left[\bigcup_{\xi\in\Xi_0} H_{w(\xi)}\right] = F$. Call an element of the partition F_x **heavy** if it contains some $H_{w(\xi)}$, $\xi \in \Xi_0$. It is easy to see that the cardinality of all heavy elements does not exceed $\mathfrak{m}$, and also that the set of points $\{x\} \subset X$ which are images of heavy elements is everywhere dense in bX and, consequently, in X. Theorem 1 is proved.

Proof of Theorem 2. By virtue of Theorem 2 ([1]), it suffices to show that $\chi(x, R) \le \mathfrak{m}$ for all $x \in R$, or, as is easy to observe, one must show that $\chi(\widetilde{F}_x, D^\tau) \le \mathfrak{m}$, if $x \in R \setminus M$, and $\widetilde{F}_x = F_x \cap [\bigcup_{x\in X} F_x]_{D^\tau}$, $F_x = F^{-1}(x)$.

Let $\{O_\alpha, \widetilde{F}_x\}$ be a fundamental system of open-closed sets of infinite cardinality forming a neighborhood base of $\widetilde{F}_x$ in D^τ. For each $O_\alpha \widetilde{F}_x$, by virtue of the continuity of the mapping and the definition of $\widetilde{F}_x$, there is a family $\{F_{x_\beta}\}$, $x_\beta \in M$, such that

$$\widetilde{F}_x \subset \left[\bigcup_\beta F_{x_\beta}\right] \subset O_\alpha \widetilde{F}_x.$$

Since $\chi(x, R) \leq \mathfrak{m},\ x \in M$, it follows that $\chi(F_x, D^\tau) \leq \mathfrak{m}$. Applying A (1), we obtain that

$$F_x = \bigcup_{\xi \in \Xi_x} H^\xi_{w_x},$$

where $\operatorname{card} w_\xi \leq \mathfrak{m},\ x \in M$. Denote

$$w = \bigcup_{x \in M} w_x$$

and

$$\Xi = \bigcup_{x \in M} \Xi_x.$$

Since $\operatorname{card} M \leq \mathfrak{m}$ and $\operatorname{card} w_x \leq \mathfrak{m}$ when $x \in M$, we have $\operatorname{card} w \leq \mathfrak{m}$. Since $w_x \subseteq w$, by virtue of 4), each layer $H^\xi_{w_x}$, $\xi \in \Xi_x$, can be represented in the form

$$H^\xi_{w_x} = \bigcup_{\eta \in \Theta_\xi} H^\eta_w.$$

Thus:

$$F_x = \bigcup_{\xi \in \Xi_x} H^\xi_{w_x} = \bigcup_{\xi \in \Xi_x} \bigcup_{\eta \in \Theta_\xi} H^\eta_w = \bigcup_{\eta \in \Theta_x} H^\eta_w, \qquad \text{where } \Theta_x = \bigcup_{\xi \in \Xi_x} \Theta_\xi.$$

Now consider some family $\{F_{x_\beta}\}$, $x_\beta \in M$. We have

$$\bigcup_\beta F_{x_\beta} = \bigcup_\beta \bigcup_{\eta \in \Theta_\beta} H^\eta_w = \bigcup_{\eta \in \Theta'} H^\eta_w, \qquad \text{where } \Theta' = \bigcup_\beta \Theta_\beta.$$

Applying Lemma 2, we obtain

$$H_\alpha = \left[\bigcup_\beta F_{x_\beta}\right] = \left[\bigcup_{\eta \in \Theta'} H^\eta_w\right] = \bigcup_{\eta \in \Theta_\alpha} H^\eta_w, \qquad \Theta_\alpha \supseteq \Theta'.$$

Thus, $\widetilde{F}_x \subseteq H_\alpha \subset O_\alpha \widetilde{F}_x$ for every α, and all

$$H_\alpha = \bigcup_{\eta \in \Theta_\alpha} H^\eta_w$$

and $\operatorname{card} w \leq \mathfrak{m}$. Note that

$$\bigcap_\alpha O_\alpha \widetilde{F}_x = \bigcap_\alpha H_\alpha = \widetilde{F}_x.$$

Hence we obtain

$$\widetilde{F}_x = \bigcap_\alpha H_\alpha = \bigcap_\alpha \left(\bigcup_{\eta \in \Theta_\alpha} H_w^\eta \right) = \bigcup_\xi \bigcap_\alpha H_w^{\eta(\alpha)}.$$

The summation over ξ ranges over all possible nonempty intersections, which are formed as follows: from each set H_α of the form

$$\bigcup_{\eta \in \Theta_\alpha} H_w^\eta$$

an arbitrary layer $H_w^{\eta(\alpha)}$ is taken, and the intersection of all the selected layers is considered. Since, by virtue of 2), two layers with one and the same base are either disjoint or coincide, we have

$$\bigcap_\alpha H_w^{\eta(\alpha)} = H_w^\xi.$$

Thus,

$$\widetilde{F}_x = \bigcup_\xi H_w^\xi,$$

where $\operatorname{card} w \leq \mathfrak{m}$. Applying now assertion D (1), we obtain that

$$\chi(\widetilde{F}_x, D^\tau) \leq \mathfrak{m}.$$

The theorem is proved.

Moscow State University
named after M. V. Lomonosov

Received
27 II 1963

References

1. B. Efimov, DAN, **149**, No. 5 (1963).

Note: Figure translations are in progress. See original paper for figures.