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Abstract

Full Text

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ON SOME CLASSES OF FUNCTIONS

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I. It is known that the Fourier transform of a function $f \in L_1(-\infty, \infty)$ is defined as follows

$$F(f) = \hat{f}(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ivx}\, dx.$$

If $f \in L_p(-\infty, \infty)$ $(1 < p \leq 2)$, then the function

$$F_R(v) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} f(x) e^{-ivx}\, dx$$

as $R \to \infty$ converges in the metric of the space $L_q(-\infty, \infty)$ to a certain function $\hat{f}(v)$, called the Fourier transform of the function $f(x)$, where $p + q = pq$ (see ([1])).

The Fourier-Stieltjes transform of a function $f(x)$ of bounded variation on $(-\infty, \infty)$, i.e., of a function $f(x) \in BV(-\infty, \infty)$, will be denoted by

$$F_S(f) = \check{f}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iux}\, df(x).$$

Let

$$\Delta_t^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x + kt)$$

be the finite difference of the function $f(x)$ of order m with step t. Put

$$\varphi_m(f, x, t) = [\Delta_t^m + \Delta_{-t}^m] f(x).$$

Theorem 1. Let $f \in L_p(-\infty, \infty)$ $(1 \leq p \leq 2)$. If

$$\lim_{t \to 0} \frac{1}{t^{2k}} \left(\int_{-\infty}^{\infty} |\varphi_{2k}(f, x, t)|^p\, dx \right)^{1/p} = 0, \tag{1}$$

$$\lim_{t\to\infty} \frac{1}{t^{2k+2}} \left(\int_{-\infty}^{\infty} |\varphi_{2k+1}(f,x,t)|^p \, dx \right)^{1/p} = 0, \tag{2}$$

then $f(x) = 0$ almost everywhere on $(-\infty, \infty)$.

Theorem 2. Let $f \in L_1(-\infty, \infty)$. Then:

1°. In order that the relation

$$\int_{-\infty}^{\infty} |\varphi_{2k}(f,x,t)| \, dx = O(t^{2k}), \tag{3}$$

hold, it is necessary and sufficient that the expression $(-1)^k v^{2k} \hat{f}(v)$ be a transfor-

the Fourier-Stieltjes transform of some function $g(x) \in BV(-\infty, \infty)$, i.e.

$$(-1)^k v^{2k} \hat{f}(v) = \check{g}(v). \tag{4}$$

2°. In order that the relation

$$\int_{-\infty}^{\infty} |\varphi_{2k+1}(f,x,t)| \, dx = O(t^{2k+2}), \tag{5}$$

hold, it is necessary and sufficient that

$$(-1)^{k+1} v^{2k+2} \hat{f}(v) = \check{g}(v), \tag{6}$$

where $g(x) \in BV(-\infty, \infty)$.

Theorem 3. Let $f \in L_p(-\infty, \infty)$ $(1 < p \le 2)$. Then

1°. In order that the relation

$$\left(\int_{-\infty}^{\infty} |\varphi_{2k}(f,x,t)|^p \, dx \right)^{1/p} = O(t^{2k}), \tag{7}$$

hold, it is necessary and sufficient that the expression $(-1)^k v^{2k} \hat{f}(v)$ be the Fourier transform of some function $g(x) \in L_p(-\infty, \infty)$, i.e.

$$(-1)^k v^{2k} \hat{f}(v) = \hat{g}(v). \tag{8}$$

2°. In order that the relation

$$\left(\int_{-\infty}^{\infty} |\varphi_{2k+1}(f,x,t)|^p\,dx\right)^{1/p} = O(t^{2k+2}), \tag{9}$$

hold, it is necessary and sufficient that

$$(-1)^{k+1} v^{2k+2} \hat{f}(v) = \hat{g}(v), \tag{10}$$

where $g(x) \in L_p(-\infty,\infty)$.

Theorems 1-3 are proved by the Fourier transform method set forth in Butzer' s papers ([3,4]). These theorems are generalizations of the corresponding results of Butzer ([3]).

It is known that if $f, g \in L_p(-\infty,\infty)$ and $(-i)^n v^n \hat{f}(v) = \hat{g}(v)$, then the function $f(x)$ has derivatives $f'(x), f''(x), \dots, f^{(n)}(x)$, and $f^{(n)}(x) \in L_p(-\infty,\infty)$ (see ([1,2])); and the relations

$$(-1)^k v^{2k} \hat{f}(v) = (-i)^{2k} v^{2k} \hat{f}(v),$$

$$(-1)^{k+1} v^{2k+2} \hat{f}(v) = (-i)^{2k+2} v^{2k+2} \hat{f}(v)$$

make it possible to characterize the class of functions $f(x)$ for which (4), (6), (8), and (10) hold, by the existence of their derivatives of the corresponding orders belonging to certain classes of functions.

II. Let us note that there are theorems analogous to Theorems 1-3 in the space $L_p(-\pi,\pi)$ ($p \geq 1$) of 2π-periodic functions (the analogue of Theorem 3 is valid for all $p > 1$).

In this case, instead of the Fourier transform of functions, one must use the Fourier coefficients of functions, and the integrals in relations (1), (2), (3), (5), (7), and (9) are replaced by the corresponding integrals taken over the interval $(-\pi,\pi)$.

III. In papers ([5]) we investigated the orders of convergence of singular integrals at those points x where the equalities

$$\int_0^h \varphi_m(f,x,t)\,dt = o(h^{\alpha+1}), \qquad \int_0^h |\varphi_m(f,x,t)|^p\,dt = o(h^{\alpha+1})$$

hold as $h \to 0$; $\alpha \geq 0$ is some nonnegative number. In addition, a number of theorems on the order of convergence of m-singular integrals in the spaces $L_p(-\infty,\infty)$ and $L_p(-\pi,\pi)$ are given there under the condition that

$$\int_0^h w(t)\,dt = o(h^{\alpha+1}),$$

where $\alpha \geq 0$ and

$$w(t) = \left(\int_{-\infty}^{\infty} |\varphi_m(f,x,t)|^p \, dx \right)^{1/p},$$

$$w = (t) \left(\int_{-\pi}^{\pi} |\varphi_m(f,x,t)|^p \, dx \right)^{1/p}$$

respectively.

The results obtained show that the study of the order of convergence of m-singular integrals in the space L_p $(p \geq 1)$ for $\alpha > m+1$ is meaningless.

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Note: Figure translations are in progress. See original paper for figures.