F. D. BERKOVICH

In the present note an infinite system of the form

1963

Abstract

Full Text

F. D. BERKOVICH

ON THE SOLUTION OF AN INFINITE SYSTEM OF LINEAR ALGEBRAIC EQUATIONS IN THE CLASS OF GROWING SEQUENCES

(Presented by Academician P. Ya. Kochina, 4 X 1962)

In the present note an infinite system of the form

$$\sum_{k=0}^{\infty} a_{n-k} f_k + \sum_{k=0}^{\infty} b_{n+k} f_k = c_n \quad (n = 0, 1, ...), \tag{1}$$

is investigated, where the free term $\{c_n\}_0^\infty$ is given, and the solution $\{f_k\}_0^\infty$ is sought in the class $m^{(s)}$ of growing sequences, whose definition is given below. The apparatus for the investigation of system (1) is the theory of boundary-value problems of analytic functions ([1], [2], [5]) and the theory of operators ([3], [4]) defined by matrices of a special form. In the case where $\{a_n\}_{-\infty}^\infty \in l_1$, $\{b_n\}_0^\infty \in l_1$, $\{c_n\}_0^\infty \in l_2$, and the solution $\{f_n\}_0^\infty$ is sought in l_2, system (1) was investigated by the author ([8]).

§ 1. Consider the space $l^{(s)}$ of sequences of complex numbers $a = \{a_n\}_{-\infty}^\infty$, absolutely convergent with weight α_n, with norm

$$\|a\|_{l^{(s)}} = \sum_{n=-\infty}^{\infty} |a_n| \alpha_n, \quad \text{where } \alpha_n = (1 + |n|)^s \ (n = 0, \pm 1, ...)^*,$$

and s is a fixed natural number. It is easy to show that $l^{(s)}$ is a normed ring ([6]). The ring $l^{(s)}$ of sequences $\{a_n\}$ is isomorphic to the ring $\mathfrak{w}^{\langle\alpha\rangle}$ of Fourier series

$$A(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

absolutely convergent with weight α_n ([6]). The isomorphism is realized by the correspondence

$$a = \{a_n\}_{-\infty}^\infty \to A(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \quad \text{and } \|A(\theta)\|_{\mathfrak{w}^{\langle\alpha\rangle}} = \|a\|_{l^{(s)}}.$$

By virtue of the choice of the weight α_n, the functions $A(\theta)$ from the ring $\mathfrak{w}^{\langle\alpha\rangle}$ will be 2π-periodic and s times continuously differentiable, and the Fourier series of the s-th derivative $A^{(s)}(\theta)$ converges absolutely.

Along with $l^{(s)}$, consider the Banach space $m^{(s)}$ of sequences $f = \{f_n\}_{-\infty}^{\infty}$ such that $f_n = O((1+|n|)^s)$ $(n = 0, \pm 1, \ldots)$, and put

$$\|f\|_{m^{(s)}} = \sup_{-\infty<n<\infty} \left\{ \frac{|f_n|}{(1+|n|)^s} \right\}.$$

It is not difficult to show that the space $m^{(s)}$ is conjugate to the ring $l^{(s)}$, $(l^{(s)})^* = m^{(s)}$**.

We shall also consider the space $(\mathfrak{w}^{\langle\alpha\rangle})^*$ of linear functionals $F(\theta)$—generalized functions (g.f.)—defined on the space of basic functions $\mathfrak{w}^{\langle\alpha\rangle}$. To each element $f = \{f_n\}_{-\infty}^{\infty}$ of the space $m^{(s)}$ we assign the g.f. $F(\theta)$ for which the numbers $\{f_n\}$ are Fourier coefficients. (It is proved that every g.f. $F(\theta)$ can be represented

---

* The results of § 3 are also valid for an arbitrary weight α_n satisfying the conditions: $\alpha_n \geq 1$, $\alpha_{n+m} \leq \alpha_n \alpha_m$

** For $s = 0$, $l^{(0)} = l_1$ coincides with the space of absolutely convergent sequences, and $m^{(0)} = m$ with the sp

uniquely by a weakly convergent Fourier series

$$F(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}.$$

§ 2. We shall consider system (1) under the assumption that $\{a_n\}$, $\{b_n\} \in l^{(s)}$, and the free term $c = \{c_n\} \in m^{(s)}$. We shall seek the solution $\{f_n\}$ in $m^{(s)}$. We shall show that, under these assumptions, system (1) is equivalent to a Carleman-type boundary-value problem ([8,9]) in the class of generalized functions $(\mathfrak{w}^{\langle\alpha\rangle})^*$.

Introduce the notation

$$f_n^+ = \begin{cases} f_n, & n \geq 0, \\ 0, & n < 0, \end{cases} \qquad f_n^- = \begin{cases} 0, & n \geq 0, \\ f_n, & n < 0, \end{cases}$$

with the aid of which system (1) can be written in the form

$$\sum_{k=-\infty}^{\infty} a_{n-k} f_k^+ + \sum_{k=-\infty}^{\infty} b_{n+k} f_k^+ = c_n + f_n^- \qquad (n = 0, \pm 1, \ldots). \tag{1'}$$

Using the uniqueness of the representation of a generalized function $F(\theta)$ by a Fourier series, Parseval' s equality, and also the relations easily verified with its aid,

$$(F(\theta), e^{-in\theta}A(\theta)) = 2\pi \sum_{k=-\infty}^{\infty} a_{n-k} f_k,$$

$$(F(-\theta), e^{-in\theta}A(\theta)) = (F(\theta), e^{in\theta}A(-\theta)) = 2\pi \sum_{k=-\infty}^{\infty} a_{n+k} f_k,$$

system (1′) can be rewritten as follows:

$$\frac{1}{2\pi}(F^+(\theta)A(\theta), e^{-in\theta}) + \frac{1}{2\pi}(F^+(-\theta)B(\theta), e^{-in\theta}) = \frac{1}{2\pi}(C(\theta) + F^-(\theta), e^{-in\theta}) \qquad (n = 0, \pm 1, ...)^*.$$

The latter relations may be regarded as equalities of the Fourier coefficients of two generalized functions $F^+(\theta)A(\theta) + F^+(-\theta)B(\theta)$ and $C(\theta) + F^-(\theta)$. Thus we arrive at a Carleman-type problem: find a function $F^+(z)$, analytic inside the unit circle S^+, and a function $F^-(z)$, analytic outside the unit circle S^- and vanishing at infinity, satisfying the boundary condition

$$A(\theta)F^+(\theta) + B(\theta)F^+(-\theta) = C(\theta) + F^-(\theta) \qquad (0 \le \theta \le 2\pi), \tag{2}$$

where $A(\theta)$, $B(\theta)$, and $C(\theta)$ are given functions, with $A(\theta)$ and $B(\theta) \in \mathfrak{w}^{\langle\alpha\rangle}$, $C(\theta) \in (\mathfrak{w}^{\langle\alpha\rangle})^*$, and the boundary values $F^\pm(\theta)$ of the functions $F^\pm(z)$ are certain generalized functions from $(\mathfrak{w}^{\langle\alpha\rangle})^*$. In this case (see (?)),

$$F^\pm(\theta) = F^\pm(e^{i\theta}) = \lim_{|z|\to 1} F^\pm(z) = \lim_{r\to 1} F^\pm(re^{i\theta}),$$

where the limit is understood in the sense of convergence in $(\mathfrak{w}^{\langle\alpha\rangle})^*$.

Starting from the boundary condition (2) and repeating the preceding arguments in reverse order, we arrive at system (1). Thus the equivalence of system (1) and problem (2) is proved. Moreover, to each solution of problem (2) there corresponds, by the formula

$$f_n = \frac{1}{2\pi}(F^+(\theta), e^{-in\theta}) \qquad (n = 0, 1, ...),$$

a certain solution of system (1).

§ 3. Consider the operators

$$A^* f = \left\{ \sum_{k=0}^{\infty} a_{n-k} f_k \right\}, \qquad B^* f = \left\{ \sum_{k=0}^{\infty} b_{n+k} f_k \right\}$$

---

*

Multiplication of a generalized function $F(\theta)$ by a basic function $B(\theta)$, as usual (see ([7])), is defined by the rule

$$(F(\theta)B(\theta), A(\theta)) = (F(\theta), A(\theta)B(\theta)).$$

$(n = 0, 1, ...)$. With a view to applying the results of [3], we shall give some definitions from it. Let M be a certain linear operator acting in the Banach space B, and let M^* be the adjoint operator; let $\alpha(M)$ and $\beta(M)$ be, respectively, the dimensions of the subspaces of solutions of the equations $M\varphi = 0$ $(\varphi \in B)$, $M^* f = 0$ $(f \in B^*)$. If the numbers $\alpha(M)$ and $\beta(M)$ are finite and the operator M is normally solvable, then M is called a Φ-operator, and the difference $\chi(M) = \alpha(M) - \beta(M)$ is called the index of the operator M.

Theorem 1. Let $\{a_n\}_{-\infty}^{\infty} \in l^{(s)}$ and $A(\theta) \neq 0$; then the operator A^* is a Φ-operator in the space $m^{(s)}$, and the index $\chi(A^*)$ of the operator A^* is computed by the formula

$$\chi(A^*) = -\operatorname{Ind}_{|t|=1} A(t) = -\frac{1}{2\pi} \int_0^{2\pi} d \ln A(\theta). \tag{3}$$

To prove Theorem 1 it is enough (see [3]) to show that the operator A,

$$A\varphi = \left\{ \sum_{k=0}^{\infty} a_{k-n} \varphi_k \right\} \quad (n = 0, 1, ...)$$

in the space $l^{(s)}$, for which A^* is adjoint, is a Φ-operator. The proof of this latter fact is based on an analogue of the Wiener-Lévy theorem for absolutely convergent Fourier series [4], proved in [6] for functions of the ring $w^{\langle \alpha \rangle}$.

Theorem 2. Let $\{b_n\}_0^{\infty} \in l^{(s)}$; then the operator B,

$$B\varphi = \left\{ \sum_{k=0}^{\infty} b_{n+k} \varphi_k \right\}$$

$(n = 0, 1, ...)$ (and consequently also B^*) is completely continuous in $l^{(s)}$ $(m^{(s)})$.

Using operator notation, we rewrite system (1) in the form

$$(A^* + B^*)f = c \qquad (c, f \in m^{(s)}).$$

Since A is a Φ-operator and B is a linear completely continuous operator, it follows (see [3]) that the operators $(A + B)$ and $(A^* + B^*)$ are also Φ-operators. Thus we arrive at the theorems:

Theorem 3. The numbers $\alpha(A + B)$ and $\beta(A + B)$ are finite.

Theorem 4. For the solvability of system (1) in the class $m^{(s)}$, it is necessary and sufficient that the conditions

$$\sum_{n=0}^{\infty} c_n g_{nj} = 0, \qquad j = 1, 2, \dots, \alpha(A + B),$$

be satisfied, where $\{g_{nj}\}$ is a collection of linearly independent solutions from $l^{(s)}$ of the homogeneous system

$$\sum_{k=0}^{\infty} a_{k-n} g_k + \sum_{k=0}^{\infty} b_{n+k} g_k = 0 \quad (n = 0, 1, \dots), \tag{4}$$

for which system (1) is adjoint.

Theorem 5. Let $m^* = \beta(A + B)$, $m = \alpha(A + B)$ be, respectively, the numbers of solutions of the homogeneous systems (1) and (4); then the formula

$$m^* - m = \chi(A^*)$$

holds, where $\chi(A^*)$ is computed by formula (3).

§ 4. Under certain additional restrictions imposed on the coefficients $\{a_n\}$ and $\{b_n\}$ of system (1), one can find its solution in closed form.

Theorem 6. Let the coefficients $\{a_n\}_{-\infty}^{\infty}$, $\{b_n\}_{-\infty}^{\infty}$ satisfy the additional conditions

$$\sum_{k=-\infty}^{\infty} a_{n+k} a_k = \sum_{k=-\infty}^{\infty} b_{n+k} b_k \quad (n = 0, \pm 1, \pm 2, \dots),$$

or, what is the same,

$$A(\theta)A(-\theta) = B(\theta)B(-\theta);$$

then problem (2) is equivalent to a pair of Carleman problems for the exterior and the interior of the unit circle (see (9)):

$$F^{-}(-\theta) = A(-\theta)B^{-1}(\theta)F^{-}(\theta) + A(-\theta)B^{-1}(\theta)C(\theta) - C(-\theta)$$
$$(0 \le \theta \le 2\pi); \tag{5}$$

$$F^{+}(\theta) = -B(\theta)A^{-1}(\theta)F^{+}(-\theta) + [F^{-}(\theta) + C(\theta)]A^{-1}(\theta)$$
$$(0 \le \theta \le 2\pi), \tag{6}$$

and the solution of system (1) is found in explicit form.

Denote $\varkappa_+ = \operatorname{Ind}\{B(\theta)A^{-1}(\theta)\}$, $\varkappa_- = \operatorname{Ind} G(e^{i\theta}) = \operatorname{Ind}\{A(-\theta)B^{-1}(\theta)\}$; by $\varkappa'_\pm$ denote the number $\frac{1}{2}\varkappa_\pm$, if $\varkappa_\pm$ is even, and $\frac{1}{2}(\varkappa_\pm + 1)$, if $\varkappa_\pm$ is odd.

Then:

1) If $\varkappa_- > 0,\ \varkappa_+ > 0$, then $m = 0,\ m^* = \varkappa(A^*)$. To this one should add the case $\varkappa_- = 0,\ G(1) = 1,\ \varkappa_+ > 0$. If, however, $\varkappa_- = 0,\ G(1) = -1,\ \varkappa_+ > 0$, then $m^* = \varkappa(A^*) + 1,\ m = 1$.

2) If $\varkappa_- < 0,\ \varkappa_+ < 0$, then $m^* = 0,\ m = |\varkappa(A^*)|$.

3) If $\varkappa_- < 0,\ \varkappa_+ > 0$, then for $G(1) = -1$

$$m^* = \varkappa'_+ + 1, \quad m = |\varkappa'_-| + 1,$$

and for $G(1) = 1$

$$m^* = \varkappa'_+, \quad m = |\varkappa'_-|.$$

4) If $\varkappa_- > 0,\ \varkappa_+ < 0$, then for $G(1) = 1$

$$m^* = \varkappa'_- - r, \quad m = |\varkappa'_+| - r,$$

and for $G(1) = -1$

$$m^* = \varkappa'_- - 1 - r_1, \quad m = |\varkappa'_+| - 1 - r_1,$$

where $r(r_1)$ is the rank of a matrix with $\varkappa'_-$ columns and $|\varkappa'_+|$ rows ($\varkappa'_- - 1$ columns and $|\varkappa'_+| - 1$ rows), whose elements are explicitly expressed in terms of $A(\theta)$ and $B(\theta)$.

We note that the solution of problems (5) and (6) in the class o.f. $(w^{\langle\alpha\rangle})^*$ is carried out by reducing them to a Riemann problem by analytic continuation of the sought function to the whole plane by the method of N. I. Muskhelishvili ((5), pp. 105-112).

§ 5. In this section we shall assume that the coefficients $\{a_n\}$ and $\{b_n\}$ of system (1) satisfy the additional conditions

$$a_n = \bar{a}_n \quad (n = 0, \pm 1, ...); \qquad |A(\theta)| > |B(\theta)| \quad (0 \le \theta \le 2\pi). \tag{7}$$

Conditions (7) make it possible to give a qualitative investigation of system (1).

Theorem 7. Suppose that conditions (7) are satisfied. Then: 1) if $\varkappa(A^*) > 0$, then $m^* = \varkappa(A^*)$, $m = 0$; 2) if $\varkappa(A^*) \leq 0$, then $m^* = 0$, $m = |\varkappa(A^*)|$.

Remark. By the same scheme one may investigate the system

$$\sum_{k=0}^{\infty} a_{n-k} f_k + \sum_{k=0}^{\infty} b_{n+k} \bar{f}_k = c_n \qquad (n = 0, 1, \ldots), \tag{8}$$

as well as continuous analogues of the infinite systems (1) and (8)—integral equations of convolution type, which find application in certain problems of mathematical physics (diffusion problems of radiation; see (10, 11)).

The author expresses gratitude to V. S. Rogozhin for a number of valuable suggestions.

Received
1 IX 1962

REFERENCES

1. F. D. Gakhov, *Boundary Value Problems*, Moscow, 1958.

2. V. S. Rogozhin, *Siberian Mathematical Journal*, 2, No. 5 (1961).

3. I. Ts. Gokhberg, M. G. Krein, *Uspekhi Mat. Nauk*, 12, issue 2 (74) (1957).

4. M. G. Krein, *Uspekhi Mat. Nauk*, 13, issue 5 (83) (1958).

5. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1946.

6. I. M. Gel' fand, D. A. Raikov, G. E. Shilov, *Commutative Normed Rings*, Moscow, 1960.

7. I. M. Gel' fand, G. E. Shilov, *Generalized Functions*, vol. 1, 1958.

8. F. D. Berkovich, All-Union Conference on the Application of Methods of the Theory of Functions of a Complex Variable to Problems of Mathematical Physics, Abstracts of Reports, Tbilisi, 1961, p. 16.

9. D. A. Kveselava, *Proceedings of the Tbilisi Mathematical Institute*, 16, 40 (1948).

10. V. V. Sobolev, *Doklady AN*, 129, No. 6 (1959).

11. V. V. Sobolev, *Doklady AN*, 136, No. 3 (1961).