MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR A. MARKOV

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR A. MARKOV

ON THE INVERSION COMPLEXITY OF A SYSTEM OF BOOLEAN FUNCTIONS

1. In the note ([2]) the concept of the inversion complexity of a system of Boolean functions was introduced. For a single Boolean function, an equality was given there expressing the inversion complexity in terms of another numerical characteristic of the function—sign-change number, which characterizes the change of values of the function on increasing sequences of Boolean vectors. For systems of several Boolean functions, I have only now been able to obtain an analogous result, which constitutes the content of the present note.

2. We shall retain the previous notation and, in addition, shall use the signs sg, $\overline{\text{sg}}$, $-$, and $[\,/\,]$ in accordance with p. 200 of the Russian translation of the well-known monograph by Kleene ([1]). Indexed letters $\varepsilon_1, \dots, \varepsilon_k$ will be used as metavariables taking the values 0 and 1. The symbol

$$\max_{\varepsilon_1,\dots,\varepsilon_j} (\ )$$

will be used to denote the greatest value of the expression in parentheses over all admissible values $\varepsilon_1, \dots, \varepsilon_j$; the symbol

$$\min_{r=1}^{j} (\ )$$

will denote the least value of the expression in parentheses when r takes values from the series $1, \dots, j$.

We say of a Boolean function f of n arguments that it is **monotone** if $f(X) \leq f(Y)$ whenever the n-dimensional Boolean vectors X and Y are such that $X < Y$. It is well known that every monotone Boolean function of n arguments can be defined by a formula in n variables containing no negation signs, i.e., by a formula of inversion complexity zero. Conversely, every formula in n variables of inversion complexity zero defines a monotone Boolean function of n arguments. Thus monotone Boolean functions may be characterized as functions of inversion complexity zero.

3. Let f be a Boolean function of n arguments, and let A and B be n-dimensional Boolean vectors. We shall say of the ordered pair of vectors A and B that it is a **break** of the function f if $A < B$, whereas $f(A) > f(B)$.

We shall say of the ordered pair of vectors A and B that it is a **break of the system of Boolean functions** $f_1, \dots, f_m$ of n arguments if it is a break of at least one of the functions f_i.

We shall say of a sequence of n-dimensional Boolean vectors $A_1, \dots, A_r$ $(r > 0)$ that it is **increasing** if $A_i < A_{i+1}$ $(1 \leq i < r)$.

Suppose we have a system of Boolean functions $f_1, \dots, f_m$ of n arguments and an increasing sequence of n-dimensional Boolean vectors $A_1, \dots, A_r$. We shall call the **fall** of the system of functions $f_1, \dots, f_m$ on the sequence $A_1, \dots, A_r$ the number of those natural numbers i from the series $1, \dots, r-1$ for which the ordered pair of vectors A_i and A_{i+1} is a break of the system of functions $f_1, \dots, f_m$.

The fall of the system of functions $f_1, \dots, f_m$ on the sequence $A_1, \dots, A_r$ can obviously be computed from the tables of values of these functions. In view of the obvious possibility of compiling a list of all increasing sequences of n-dimensional Boolean vectors, the maximum fall of the given system of Boolean functions $f_1, \dots, f_m$ of n arguments over all possible increasing sequences of n-dimensional Boolean vectors can be found. We shall call this maximum the **fall of the system of functions** $f_1, \dots, f_m$ and denote it by the symbol

$$\mathrm{Des}(f_1, \dots, f_m).$$

It is clear that

$$0 \leqslant \mathrm{Des}(f_1, \dots, f_m) \leqslant n$$

for every system of Boolean functions $f_1, \dots, f_m$ of n arguments.

4. The following theorem expresses the inversion complexity of a system of Boolean functions in terms of the fall of this system.

4.1. For every system of Boolean functions of n arguments $f_1, \dots, f_m$, the equality

$$\mathrm{Inv}(f_1, \dots, f_m) = \mathrm{D}(\mathrm{Des}(f_1, \dots, f_m))$$

holds.

5. The proof of Theorem 4.1 can be based on the following lemmas.

5.1. Let $f_1, \dots, f_m$ be Boolean functions of n arguments; let $\Phi(X)$, where X is an arbitrary n-dimensional Boolean vector, denote the maximum fall of the system of functions $f_1, \dots, f_m$ on those increasing sequences of n-dimensional Boolean vectors $A_1, \dots, A_r$ for which $A_r = X$; let

$$k = \mathrm{D}(\mathrm{Des}(f_1, \dots, f_m)).$$

For every k-term tuple $\varepsilon_1, \dots, \varepsilon_k$ ($\varepsilon_i = 0, 1$ for $i = 1, \dots, k$), define the Boolean function of n arguments $t_{\varepsilon_1,\dots,\varepsilon_k}$ by the equality

$$t_{\varepsilon_1,\dots,\varepsilon_k}(X) = \mathrm{sg}\left((\Phi(X)+1) - \sum_{j=1}^{k} \varepsilon_j 2^{k-j}\right).$$

Define successively the Boolean functions h_j ($j = 1, \dots, k$) of n arguments by the equalities:

$$h_1(X) = \overline{\mathrm{sg}}(t_{1,0,\dots,0}(X)),$$

$$h_j(X) = \overline{\mathrm{sg}}\left(\max_{\varepsilon_1,\dots,\varepsilon_{j-1}}\left(\min(t_{\varepsilon_1,\dots,\varepsilon_{j-1},1,0,\dots,0}(X), \min_{r=1}^{j-1}(\max(h_r(X), \varepsilon_r)))\right)\right)$$

$$(1 < j \leqslant k).$$

There can be constructed monotone Boolean functions e_i ($1 \leqslant i \leqslant m$) of $n + k$ arguments such that the equalities

$$e_i(X, h_1(X), \dots, h_k(X)) = f_i(X) \qquad (1 \leqslant i \leqslant m)$$

hold for every Boolean vector X.

5.2. Let a system of formulas $P_1, \dots, P_m$ in n variables define a system of Boolean functions of n arguments $f_1, \dots, f_m$; let $H_1, \dots, H_k$ be the list of all negative subformulas of the system $P_1, \dots, P_m$, arranged in such a way that the formula H_j is not a subformula of the formula H_i whenever $i < j$; let h_j be the Boolean function of n arguments defined by the formula H_j ($1 \leqslant j \leqslant k$).

Define the arithmetic function ψ of an n-dimensional Boolean vector by the equality

$$\psi(X) = \sum_{j=1}^{k} h_j(X) 2^{k-j}.$$

The function ψ has the following properties:

$$\psi(X) \geqslant \psi(Y),$$

provided that $X \prec Y$;

$$\psi(X) > \psi(Y),$$

provided that the ordered pair of Boolean vectors X and Y is a drop of the system of functions $f_1, \dots, f_m$.

It follows from Lemma 5.1 that every system of Boolean functions $f_1, \dots, f_m$ in n arguments can be represented by a system of formulas in n variables whose inversion complexity is at most $D(\mathrm{Des}(f_1, \dots, f_m))$. From Lemma 5.2 it is easy to conclude that the inversion complexity of any system of formulas in n variables defining the given system of Boolean functions in n arguments $f_1, \dots, f_m$ is not less than $D(\mathrm{Des}(f_1, \dots, f_m))$. Theorem 4.1 follows from the last two propositions.

6. A system of Boolean functions, in particular, may consist of a single function. The definition of a drop, of course, also applies to this case, i.e., the expression $\mathrm{Des}(f)$ is meaningful, where f is any Boolean function.

It is not difficult, further, to find the following relation between the drop and the previously defined sign-change count of a Boolean function [(2)]

$$\mathrm{Des}(f) = [\mathrm{Alt}(f)/2].$$

Using this equality, it is easy to obtain Theorem 5.1 of note [(2)] as a consequence of Theorem 4.1. The remaining results of note [(2)]–Theorems 6.1 and 6.2, concerning the function I, are then also obtained easily.

Submitted
11 II 1963

REFERENCES

[1] S. K. Kleene, *Introduction to Metamathematics*, Moscow, 1957. [2] A. A. Markov, DAN, **116**, No. 6, 917 (1957).

Note: Figure translations are in progress. See original paper for figures.